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Spectral asymptotics for Laplacians on self-similar sets☆

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Abstract

Given a self-similar Dirichlet form on a self-similar set, we first give an estimate on the asymptotic order of the associated eigenvalue counting function in terms of a 'geometric counting function' defined through a family of coverings of the self-similar set naturally associated with the Dirichlet space.

Secondly, under (sub-)Gaussian heat kernel upper bound, we prove a detailed short time asymptotic behavior of the partition function, which is the Laplace-Stieltjes transform of the eigenvalue counting function associated with the Dirichlet form. This result can be applicable to a class of infinitely ramified self-similar sets including generalized Sierpinski carpets, and is an extension of the result given recently by B. M. Hambly for the Brownian motion on generalized Sierpinski carpets. Moreover, we also provide a sharp remainder estimate for the short time asymptotic behavior of the partition function.

Key words: self-similar sets, Dirichlet forms, eigenvalue counting function, partition function, short time asymptotics, sub-Gaussian heat kernel estimate, Sierpinski carpets
2000 MSC: 28A80, 35P20, 31C25, 60J45, 49R50

Contents

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1. Introduction

Mathematical analysis on fractal spaces began when Goldstein [19] and Kusuoka [31] had constructed the *Brownian motion* on the *Sierpinski gasket* (Figure 1.1 below), whose transition density (heat kernel) has proved to be subject to the two-sided sub-Gaussian estimate by the result of Barlow and Perkins [8]. Since then many results have been obtained concerning the spectra of Laplacians on self-similar sets. For example, let $\{\lambda_n^{\mathrm{SG}}\}_{n\in\mathbb{N}}$ be the non-decreasing enumeration of the eigenvalues of the Laplacian associated with the Brownian motion on the Sierpinski gasket, where each eigenvalue is repeated according to its multiplicity. The corresponding *eigenvalue counting function* is defined by

$$N_{\mathrm{SG}}(x) := \#\{n \in \mathbb{N} \mid \lambda_n^{\mathrm{SG}} \leq x\} \tag{1.1}$$

for each $x \in [0,\infty)$, where $\#A$ denotes the number of all the elements of a set A. By the results of Fukushima and Shima [18], Kigami and Lapidus [30] and Barlow and Kigami [10], there exists a $\log 5$-periodic right-continuous *discontinuous* function $G : \mathbb{R} \to (0,\infty)$ with $0 < \inf_{\mathbb{R}} G < \sup_{\mathbb{R}} G < \infty$, such that

$$N_{\mathrm{SG}}(x) = x^{d_{\mathrm{S}}/2} G(\log x) + O(1) \tag{1.2}$$

as $x \to \infty$, where $d_{\mathrm{S}} := \log 9/\log 5$.

This result is in remarkable contrast to Weyl's theorem [35, 36] for the Dirichlet Laplacian on bounded open subsets of Euclidean spaces in two important points, as suggested in the early 1980s by Physicists, e.g. Rammal and Toulouse [34] and Rammal [33]. First, the ratio $x^{-d_{\mathrm{S}}/2} N_{\mathrm{SG}}(x)$ is bounded away from 0 and ∞ but does not converge as $x \to \infty$. Secondly, the number d_{S}, called the *spectral dimension* of the Sierpinski gasket, is different from its Box-counting dimension (and the Hausdorff dimension) $d_f = \log 3/\log 2$ with respect to the Euclidean distance; $d_{\mathrm{S}} < d_f$. By [30, 10], the same kind

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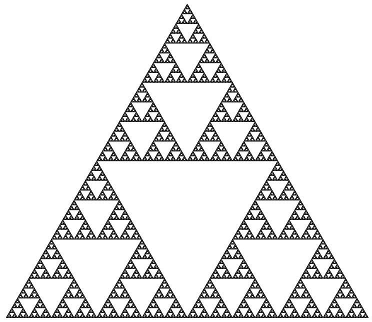

Figure 1.1: *The Sierpinski gasket*

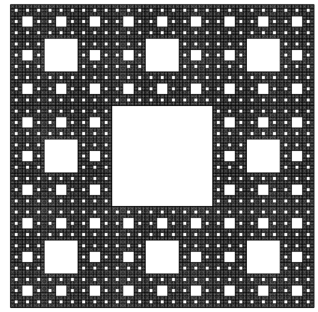

Figure 1.2: *The Sierpinski carpet*

of result is known to be valid for *nested fractals*, a class of finitely ramified self-similar sets.

The purpose of this paper is twofold. First, we give a geometric characterization of the spectral dimension d_{S} based on a framework due to Kigami [28]. Secondly, we prove the same kind of asymptotic behavior as in (1.2) of the *partition function*, the Laplace-Stieltjes transform of the eigenvalue counting function, for the case of infinitely ramified self-similar sets such as the *Sierpinski carpet* (Figure 1.2). All our results are applicable to a class of infinitely ramified self-similar sets including *generalized Sierpinski carpets* (see [6, 7]), but in this introduction we illustrate the main results by treating the case of the Sierpinski carpet as a particular example.

Let $\{F_i\}_{i\in S}$, $S := \{1,\dots,8\}$, be a family of similitudes on $\mathbb{R}^2$ as described in Figure 1.3 below, where the whole square denotes $[0,1]^2$. The Sierpinski carpet K is defined as the self-similar set associated with $\{F_i\}_{i\in S}$, that is, the unique non-empty compact subset of $\mathbb{R}^2$ such that $K = \bigcup_{i\in S} F_i(K)$. Let $V_0 := [0,1]^2 \setminus (0,1)^2$, which should be regarded as the *boundary of* K: In fact, V_0 is the smallest subset of K that satisfies $F_i(K)\cap F_j(K) = F_i(V_0)\cap F_j(V_0)$ for any distinct $i,j\in S$. As $\#V_0 = \infty$, K is *infinitely ramified*.

Let ν be the self-similar measure with weight $(1/8,\dots,1/8)$. By the results of Barlow and Bass [1, 2, 3, 4] and Kusuoka and Zhou [32, Section 8], there exists a regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(K,\nu)$ satisfying $\mathcal{F}\subset\{u \mid u : K\to\mathbb{R}, u \text{ is continuous}\}(=: C(K))$ and such that

$$\mathcal{E}(u,v) = \sum_{i\in S}\frac{1}{r}\mathcal{E}(u\circ F_i, v\circ F_i), \qquad u,v\in\mathcal{F} \tag{1.3}$$

for some $r\in(0,1)$ (note also the recent result [7] on uniqueness of such $(\mathcal{E},\mathcal{F})$). Moreover, by looking at [32, Theorems 4.5, 5.4, 6.9 and 7.2], we easily verify that $(\mathcal{E},\mathcal{F})$ is a resistance form on K whose associated resistance metric is compatible with the original (Euclidean) topology of K. (See [27, Chapter 2] and [29, Part I] for basic theory of resistance forms.) Let μ be a Borel probability measure on K which is *elliptic*, i.e. there exists $\gamma\in(0,\infty)$ such that $\mu(K_{wi})\geq\gamma\mu(K_w)$ for any $w\in\bigcup_{m\in\mathbb{N}\cup\{0\}} S^m(=: W_*)$ and any $i\in S$, where $F_w := F_{w_1}\circ\cdots\circ F_{w_m}$ and $K_w := F_w(K)$ for $w = w_1\dots w_m\in W_*$. Then by [29, Corollary 5.4 and Theorem 8.4], $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form on $L^2(K,\mu)$. Also, (1.3) implies the strong locality of $(\mathcal{E},\mathcal{F})$. This Dirichlet space $(\mathcal{L} := (K,S,\{F_i\}_{i\in S}),\mu,\mathcal{E},\mathcal{F},r)$ is the framework of our study.

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$\boldsymbol{F_7}$	$\boldsymbol{F_6}$	$\boldsymbol{F_5}$
$\boldsymbol{F_8}$		$\boldsymbol{F_4}$
$\boldsymbol{F_1}$	$\boldsymbol{F_2}$	$\boldsymbol{F_3}$

Figure 1.3: *The similitudes* $\{F_i\}_{i\in S}$

$\boldsymbol{\mu_7}$	$\boldsymbol{\mu_6}$	$\boldsymbol{\mu_5}$
$\boldsymbol{\mu_8}$		$\boldsymbol{\mu_4}$
$\boldsymbol{\mu_1}$	$\boldsymbol{\mu_2}$	$\boldsymbol{\mu_3}$

Figure 1.4: *The weight* $(\mu_i)_{i\in S}$

To explain our first main result, let us define several notions concerning the description of the geometry of the space $(\mathcal{L},\mu,\mathcal{E},\mathcal{F},r)$. Let $|w|:=m$ for $w=w_1\ldots w_m\in S^m$, $m\in\mathbb{N}\cup\{0\}$. Set $g(w):=\sqrt{r^{|w|}\mu(K_w)}$ for $w\in W_*$ and define

$$\Lambda_s:=\{w_1\ldots w_m\in W_*\mid g(w_1\ldots w_{m-1})>s\geq g(w_1\ldots w_m)\} \tag{1.4}$$

for $s\in(0,1]$, with the convention that $g(w_1\ldots w_{m-1})=2$ when $m=0$. g is called the *gauge function* and the collection $\mathcal{S}:=\{\Lambda_s\}_{s\in(0,1]}$ is called the *scale*, respectively, *associated with the Dirichlet space* $(\mathcal{L},\mu,\mathcal{E},\mathcal{F},r)$. We regard each K_w, $w\in\Lambda_s$ (or strictly speaking, the union $K^{(0)}(\Lambda_s,K_w):=\bigcup\{K_v\mid v\in\Lambda_s, K_v\cap K_w\neq\emptyset\}$) as a ball of radius s. There may *not* be an associated distance, but under certain conditions we can associate a *qdistance* d *adapted to* $\mathcal{S}$ (see Subsection 2.4 below and [28, Section 2.3]) so that, for some $c_1,c_2\in(0,\infty)$, each $K^{(0)}(\Lambda_s,K_w)$, $s\in(0,1]$, $w\in\Lambda_s$, is comparable to metric balls with respect to d of radii c_1s and c_2s. It is clear that $K=\bigcup_{w\in\Lambda_s}K_w$. Also for distinct $w,v\in\Lambda_s$, we see that $K_w\cap K_v=F_w(V_0)\cap F_v(V_0)$, that is, K_w and K_v intersect only on their boundaries. In this sense, $\{K_w\mid w\in\Lambda_s\}$ may be thought of as a covering of K by 'balls of radius s' with small overlaps. Now our first main theorem (Theorem 4.3) together with Proposition 4.4 yields the following statement. Let $\mathcal{F}_0:=\{u\in\mathcal{F}\mid u|_{V_0}=0\}$ and let H_N (resp. H_D) be the non-negative self-adjoint operator on $L^2(K,\mu)$ associated with $(\mathcal{E},\mathcal{F})$ (resp. $(\mathcal{E}|_{\mathcal{F}_0\times\mathcal{F}_0},\mathcal{F}_0)$).

Theorem 1.1 *Let N_N (resp. N_D) be the eigenvalue counting function of H_N (resp. H_D). Then there exist $c_1,c_2\in(0,\infty)$ and $\delta\in[1,\infty)$ such that for any $x\in[\delta,\infty)$,*

$$c_1\#\Lambda_{x^{-1/2}}\leq N_D(x)\leq N_N(x)\leq c_2\#\Lambda_{x^{-1/2}}. \tag{1.5}$$

Note that H_N and H_D have compact resolvents by [29, Lemma 8.6] (we will give a direct proof of this fact in Section 4). Hence N_N and N_D can be defined in the present situation.

The important point about Theorem 1.1 is the generality of the measure μ: *The only assumption on μ is that it is elliptic*, and in particular μ need **not** be a self-similar measure. With such a weak assumption, we have a geometric description (1.5) of the asymptotic order of $N_N(x)$ and $N_D(x)$ as $x\to\infty$. On the other hand, if μ is a self-similar measure on K with weight $(\mu_i)_{i\in S}$, then we can easily show the following estimate of $\#\Lambda_s$:

$$s^{-d_{\mathrm{S}}}\leq\#\Lambda_s\leq\Gamma s^{-d_{\mathrm{S}}},\qquad s\in(0,1], \tag{1.6}$$

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where $d_{\mathrm{S}}\,(\in (0,\infty))$ is the unique $d \in \mathbb{R}$ that satisfies $\sum_{i\in S}(r\mu_i)^{d/2} = 1$ and $\Gamma := (\min_{i\in S}\gamma_i)^{-d_{\mathrm{S}}}$. By (1.5) and (1.6), we may call d_{S} the *spectral dimension of the Dirichlet space* $(\mathcal{L},\mu,\mathcal{E},\mathcal{F},r)$, and we have a geometric characterization (1.6) of d_{S}.

Next we turn to the second purpose of this paper. In the rest of this introduction, μ is assumed to be a self-similar measure on K with weight $(\mu_i)_{i\in S} \in (0,1)^S$, $\sum_{i\in S}\mu_i = 1$. Unfortunately, it seems extremely difficult to verify directly an asymptotic behavior similar to (1.2) of N_b for $b \in \{N,D\}$ in the present case, as K is infinitely ramified. But since it may be possible to make use of arguments on the corresponding diffusion process and heat kernel estimates, there is some hope of proving a result similar to (1.2) for the associated *partition function* $Z_b : (0,\infty) \to (0,\infty)$ defined by

$$Z_b(t) := \mathrm{Tr}(e^{-tH_b}) = \sum_{n\in\mathbb{N}} e^{-t\lambda_n^b} = \int_{[0,\infty)} e^{-ts}\,dN_b(s), \tag{1.7}$$

where $\{\lambda_n^b\}_{n\in\mathbb{N}}$ is the non-decreasing enumeration of the eigenvalues of H_b, $b \in \{N,D\}$. In fact, our second main result (Theorem 5.2) and its corollary (Corollary 5.4) lead us to the following Theorem. Let $\gamma_i := \sqrt{r\mu_i}$ for $i \in S$ and let d_{S} be as in (1.6).

Theorem 1.2 *Assume the following condition on $(\mu_i)_{i\in S}$ (see Figure 1.4 above):*

$$\mu_1 = \mu_3 = \mu_5 = \mu_7, \quad \mu_2 = \mu_6 \ \text{ and } \ \mu_4 = \mu_8. \tag{1.8}$$

Then we have the following statements.
(1) Non-lattice case: If $\sum_{i\in S}\mathbb{Z}\log\gamma_i$ is a dense additive subgroup of $\mathbb{R}$, then for $b \in \{N,D\}$, $t^{d_{\mathrm{S}}/2}Z_b(t)$ converges as $t \downarrow 0$, so does $x^{-d_{\mathrm{S}}/2}N_b(x)$ as $x \to \infty$ and

$$\lim_{t\downarrow 0} t^{d_{\mathrm{S}}/2}Z_N(t) = \lim_{t\downarrow 0} t^{d_{\mathrm{S}}/2}Z_D(t) \in (0,\infty), \tag{1.9}$$

$$\lim_{x\to\infty}\frac{N_N(x)}{x^{d_{\mathrm{S}}/2}} = \lim_{x\to\infty}\frac{N_D(x)}{x^{d_{\mathrm{S}}/2}} \in (0,\infty). \tag{1.10}$$

(2) Lattice case: If $\sum_{i\in S}\mathbb{Z}\log\gamma_i$ is a discrete additive subgroup of $\mathbb{R}$ with generator $T \in (0,\infty)$, then there exists a continuous T-periodic function $G : \mathbb{R} \to (0,\infty)$ such that, for $b \in \{N,D\}$,

$$\lim_{t\downarrow 0}\Big[t^{d_{\mathrm{S}}/2}Z_b(t) - G\Big(\frac{1}{2}\log\frac{1}{t}\Big)\Big] = 0. \tag{1.11}$$

This theorem is an extension of Hambly's recent result [21, Theorem 1.1], which concentrates on the case where $\mu_i = 1/8$ for any $i \in S$. The reason for the condition (1.8) is that, by [28, Theorems 3.2.3 and 3.4.5], it is equivalent to the following *(sub-)Gaussian heat kernel upper bound* (UHK): With some $\beta \in (1,\infty)$ and a distance d on K which is *'adapted to the scale $\mathcal{S}$'*, for any $(t,x,y) \in (0,1]\times K\times K$,

$$p_t^N(x,y) \le \frac{c_1}{\mu(B_{t^{1/\beta}}(x,d))}\exp\bigg(-c_2\Big(\frac{d(x,y)^\beta}{t}\Big)^{\frac{1}{\beta-1}}\bigg), \tag{UHK}$$

where $\{p_t^N\}_{t\in(0,\infty)}$ is the (unique) jointly continuous heat kernel of $\{e^{-tH_N}\}_{t\in(0,\infty)}$ and $B_r(x,d) := \{y \in K \mid d(y,x) < r\}$. (See [29, Theorem 9.4] for existence and continuity of the heat kernel, and Definition 5.1 for the precise statement of (UHK).) Note that in

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(UHK) we allow the cases with strong **spatial inhomogeneity**: Unless $\mu_i = 1/8$ for any $i \in S$, $\limsup_{t\downarrow 0}\bigl(\log \mu(B_{t^{1/\beta}}(x,d))\bigr)/\log t^{-1}$ and $\liminf_{t\downarrow 0}\bigl(\log \mu(B_{t^{1/\beta}}(x,d))\bigr)/\log t^{-1}$ depend highly on $x \in K$.

The key part of the proof of Theorem 1.2 is to prove that the difference $Z_N - Z_D$ is sufficiently smaller, compared with Z_N and Z_D. In fact, we have the following estimate.

Theorem 1.3 *Assume* (1.8). *Choose $d_\partial \in (0,\infty)$ so that $2\gamma_1^{d_\partial} + (\max\{\gamma_2,\gamma_4\})^{d_\partial} = 1$. Then there exists $c_3, c_4 \in (0,\infty)$ such that for any $t \in (0,1]$,*

$$c_3 t^{-d_\partial/2} \le Z_N(t) - Z_D(t) \le c_4 t^{-d_\partial/2}. \tag{1.12}$$

Note that d_∂ admits the following estimate; there exists $c_5, c_6 \in (0,\infty)$ such that

$$c_5 s^{-d_\partial} \le \#(\{w \in \Lambda_s \mid K_w \cap V_0 \ne \emptyset\}) \le c_6 s^{-d_\partial}, \qquad s \in (0,1]. \tag{1.13}$$

In this sense we will call d_∂ the *cell-counting dimension of V_0 with respect to the scale $\mathcal{S}$*.

Since we have a trivial lower bound $Z_N(t) - Z_D(t) \ge 0$, $t \in (0,\infty)$, the upper inequality of (1.12) suffices for the proof of Theorem 1.2, and it is a special case of Theorem 5.11. Note that the lower bound in (1.12) is *new even when $\mu_i = 1/8$ for any $i \in S$*, and essentially as its corollary, the following sharp remainder estimate also follows.

Theorem 1.4 *Suppose $\mu_i = 1/8$ for any $i \in S$ and let $G : \mathbb{R} \to (0,\infty)$ be as in Theorem 1.2 (2). Then there exist $c_7, c_8 \in (0,\infty)$ such that for any $t \in (0,1]$,*

$$c_7 t^{-d_\partial/2} \le t^{-d_{\mathcal{S}}/2} G\Bigl(\frac{1}{2}\log\frac{1}{t}\Bigr) - Z_D(t) \le c_8 t^{-d_\partial/2}. \tag{1.14}$$

Theorem 1.3 is a special case of *Theorem 7.7, which may be seen as the third main result of this article*. In fact, Theorem 7.7 treats the similar lower bound for the case with Dirichlet (killing) condition on a general self-similar subset of positive capacity.

Finally, we remark that almost all the arguments illustrated so far apply also to any generalized Sierpinski carpet, which has been defined in [6, 7]. See Section 8 for details.

The organization of this paper is as follows. In Section 2, we introduce a number of notions, including that of scales and gauge functions, to describe geometry of self-similar sets. In Section 3, we introduce the notion of self-similar Dirichlet spaces as the framework of our spectral analysis. We show our first main result (Theorem 4.3) in Section 4. Section 5 is devoted to the statement and the proof of our second main theorem (Theorem 5.2) on an asymptotic expansion of the partition function. The key for Theorem 5.2 is Theorem 5.11, where the sub-Gaussian heat kernel upper bound plays a crucial role. As a complement to the results of Section 5, in Section 6 we provide a practical method of calculating the cell-counting dimension of the boundary of self-similar sets. In Section 7, we state and prove our *'third main theorem'* Theorem 7.7, asserting the sharpness as in (1.12) of the order estimate of the partition functions given in Theorem 5.11. In Section 8, we apply the results of the previous sections to generalized Sierpinski carpets. Then the paper is concluded by mentioning related open problems. Finally, the appendix provides a few easy but important facts playing essential roles in Section 7, which are not suitable to be included in the main text.

Notation. Throughout this paper, we follow the following notations and conventions.
(1) $\mathbb{N} = \{1,2,3,\dots\}$, i.e. $0 \notin \mathbb{N}$.

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(2) Given a topological space E, let $\mathcal{B}(E)$ denote the Borel σ-field of E. A measure μ defined on the measurable space $(E,\mathcal{B}(E))$ is called a Borel measure on E. For $f : E \to \mathbb{R}$, we write $\|f\|_\infty := \sup_{x\in E}|f(x)|$ and $\mathrm{supp}_E[f] := \overline{\{x \in E \mid f(x) \neq 0\}}$. We also write $C(E) := \{f \mid f : E \to \mathbb{R}, f \text{ is continuous}\}$, $C_b(E) := \{f \mid f \in C(E), \|f\|_\infty < \infty\}$ and $C_\infty(E) := \bigl\{f \mid f \in C(E), \{x \in E \mid |f(x)| \geq \delta\} \text{ is compact for any } \delta \in (0,\infty)\bigr\}$. Moreover, for $A \subset E$, $\mathrm{int}_E A$ denotes the interior of A in E.

2. Basics on self-similar sets

In this section, we review basic notions on self-similar sets. See Kigami [28, Sections 1.1, 1.2, 1.3 and 2.3] for details and proofs.

2.1. Scales on the shift space

First we define the notion of scales on the shift space and state their basic properties.

Definition 2.1 (Words and shift space) Let S be a non-empty finite set.
(1) We define $W_m(S) := S^m := \{w_1 \dots w_m \mid w_i \in S \text{ for } i = 1,\dots,m\}$ for $m \in \mathbb{N}$, and $W_0(S) := \{\emptyset\}$, where $\emptyset$ is an element called the *empty word*. We also set $W_\#(S) := \bigcup_{m\in\mathbb{N}} W_m(S)$ and $W_*(S) := W_\#(S) \cup \{\emptyset\}$. For $w \in W_*(S)$, the *length of* w, which is denoted by $|w|$, is defined to be the unique $m \in \mathbb{N}\cup\{0\}$ satisfying $w \in W_m(S)$.
(2) For $w = w_1 \dots w_m \in W_*(S)$, $v = v_1 \dots v_n \in W_*(S)$, we set $wv := w_1 \dots w_m v_1 \dots v_n$. Also for $w^1, w^2 \in W_*(S)$, we define

$$\begin{aligned} &w^1 \leq w^2 \text{ if and only if } w^1 = w^2 v \text{ for some } v \in W_*(S), \text{ and}\\ &w^1 < w^2 \text{ if and only if } w^1 \leq w^2 \text{ and } w^1 \neq w^2. \end{aligned}$$

(3) For $w = w_1 \dots w_m \in W_\#(S)$, we write $w_{[-1]} := w_1 \dots w_{m-1}$.
(4) The *(one-sided) shift space with symbols* S is defined by

$$\Sigma(S) := S^{\mathbb{N}} := \{\omega = \omega_1\omega_2\omega_3\cdots \mid \omega_i \in S \text{ for any } i \in \mathbb{N}\}.$$

For each $i \in S$, we define $\sigma_i : \Sigma(S) \to \Sigma(S)$ by $\sigma_i(\omega_1\omega_2\omega_3\dots) := i\omega_1\omega_2\omega_3\dots$. We also define $\sigma : \Sigma(S) \to \Sigma(S)$ by $\sigma(\omega_1\omega_2\omega_3\dots) := \omega_2\omega_3\omega_4\dots$. For $w = w_1 \dots w_m \in W_*(S)$, we write $\sigma_w := \sigma_{w_1}\circ\dots\circ\sigma_{w_m}$ and $\Sigma_w(S) := \sigma_w(\Sigma(S))$.

Note that $\leq$ is a partial order on $W_*(S)$.

We fix a non-empty finite set S in the rest of this subsection. We will write W_m, W_*, Σ and so forth instead of $W_m(S)$, $W_*(S)$ and $\Sigma(S)$ when no confusion can occur.

We consider Σ to be a topological space with the product topology inherited from the discrete topology of S. With this topology, Σ is a compact metrizable space.

Definition 2.2 (Partitions) (1) Let Λ be a finite subset of W_*. We call Λ a *partition of* Σ if and only if $\Sigma_w \cap \Sigma_v = \emptyset$ for $w, v \in \Lambda$ with $w \neq v$, and $\Sigma = \bigcup_{w\in\Lambda}\Sigma_w$.
(2) Let Λ_1 and Λ_2 be two partitions of Σ. Then we say that Λ_1 is a *refinement of* Λ_2, and write $\Lambda_1 \leq \Lambda_2$, if and only if each $w^1 \in \Lambda_1$ admits an element $w^2 \in \Lambda_2$ such that $w^1 \leq w^2$.

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Note that the relation $\leq$, which is defined on the collection of all partitions of Σ, is a partial order. Note also that, for $w, v \in W_*$, $\Sigma_w \cap \Sigma_v \neq \emptyset$ if and only if either $w \leq v$ or $v \leq w$.

Let Λ_1 and Λ_2 be partitions of Σ with $\Lambda_1 \leq \Lambda_2$. Then for any $w^1 \in \Lambda_1$, there exists a unique $w^2 \in \Lambda_2$ such that $w^1 \leq w^2$. Therefore we can naturally define a mapping $\Lambda_1 \to \Lambda_2$ by $w^1 \mapsto w^2$, with w^1 and w^2 as above. This mapping is surjective, hence $\#\Lambda_1 \geq \#\Lambda_2$, where $\#A$ denotes the number of the elements of a set A.

Definition 2.3 (Scales) Let Λ_s be a partition of Σ for any $s \in (0,1]$. Then the family $\mathcal{S} := \{\Lambda_s\}_{s\in(0,1]}$ of partitions of Σ is called a *scale on* Σ if and only if $\mathcal{S}$ satisfies the following three properties:
(S1) $\Lambda_1 = W_0$. $\Lambda_{s_1} \leq \Lambda_{s_2}$ for any $s_1, s_2 \in (0,1]$ with $s_1 \leq s_2$.
(S2) $\min\{|w| \mid w \in \Lambda_s\} \to \infty$ as $s \downarrow 0$.
(Sr) For any $s \in (0,1)$ there exists $\varepsilon \in (0, 1-s]$ such that $\Lambda_{s'} = \Lambda_s$ for any $s' \in (s, s+\varepsilon)$.

Remark. In Kigami [28], a family $\mathcal{S} = \{\Lambda_s\}_{s\in(0,1]}$ of partitions satisfying (S1) and (S2) is called a scale on Σ, and $\mathcal{S}$ is called *right-continuous* if $\mathcal{S}$ satisfies (Sr) in addition. But since we use only right-continuous scales (in the sense of [28]), we simply call them scales.

Definition 2.4 (Gauge functions) A function $g : W_* \to (0,1]$ is called a *gauge function on* W_* if and only if g has the following two properties:
(G1) $g(wi) \leq g(w)$ for any $w \in W_*$ and any $i \in S$.
(G2) $\max\{g(w) \mid w \in W_m\} \to 0$ as $m \to \infty$.

There is a natural bijection between the collection of all scales on Σ and that of all gauge functions on W_*, as in the following theorem.

Theorem 2.5 *(1) Let g be a gauge function on W_*. For each $s \in (0,1]$, define*

$$\Lambda_s(g) := \{w \in W_* \mid g(w_{[-1]}) > s \geq g(w)\}, \tag{2.1}$$

with the convention that $g(w_{[-1]}) = 2$ when $w = \emptyset$. We also set $\mathcal{S}(g) := \{\Lambda_s(g)\}_{s\in(0,1]}$. Then $\mathcal{S}(g)$ is a scale on Σ. We call $\mathcal{S}(g)$ the scale induced by the gauge function g.
(2) Let $\mathcal{S} = \{\Lambda_s\}_{s\in(0,1]}$ be a scale on Σ. Then there exists a unique gauge function $l_{\mathcal{S}}$ on W_* such that $\mathcal{S} = \mathcal{S}(l_{\mathcal{S}})$. We call $l_{\mathcal{S}}$ the gauge function of the scale $\mathcal{S}$.

By this theorem, we can identify a scale on Σ with its gauge function.

Next we define some regularity conditions for scales.

Definition 2.6 (Elliptic scales) Let $\mathcal{S} = \{\Lambda_s\}_{s\in(0,1]}$ be a scale on Σ and l be its gauge function. We consider the following two conditions on $\mathcal{S}$:
(EL1) There exists $\beta_1 \in (0,1)$ such that $l(wi) \geq \beta_1 l(w)$ for any $w \in W_*$ and any $i \in S$.
(EL2) There exist $\beta_2 \in (0,1)$ and $k \in \mathbb{N}$ such that $l(wv) \leq \beta_2 l(w)$ for any $w \in W_*$ and any $v \in W_k$.

$\mathcal{S}$ is called *elliptic* if and only if its gauge function l satisfies both (EL1) and (EL2).

The following proposition, which asserts a doubling property of the function $(0,1] \ni s \mapsto \#\Lambda_s$ for a scale $\{\Lambda_s\}_{s\in(0,1]}$, is fundamental for the results in Section 4.

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Proposition 2.7 *Let $\mathcal{S} = \{\Lambda_s\}_{s\in(0,1]}$ be a scale on Σ whose gauge function l satisfies* (EL2) *and let $\beta_2 \in (0,1)$ and $k \in \mathbb{N}$ be as in* (EL2). *Then $\#\Lambda_{\beta_2 s} \leq (\#S)^k \#\Lambda_s$ and $\#\Lambda_s \leq (\#\Lambda_{\beta_2}) s^{-\alpha}$ for any $s \in (0,1]$, where $\alpha := -(k \log \#S)/\log\beta_2 \ (\in [0,\infty))$.*

Proof. Let $s \in (0,1]$. For any $w \in \Lambda_s$ and any $v \in W_k$, we have $l(wv) \leq \beta_2 l(w) \leq \beta_2 s$ by (EL2) and Theorem 2.5. Therefore there is a unique $\tau \in \Lambda_{\beta_2 s}$ such that $wv \leq \tau$. Thus we can define a mapping $\eta : \Lambda_s \times W_k \to \Lambda_{\beta_2 s}$ by $\eta(w,v) := \tau$, with w, v, τ as above.

Let $\tau \in \Lambda_{\beta_2 s}$. Since $\Lambda_{\beta_2 s} \leq \Lambda_s$ we can choose $w \in \Lambda_s$ and $v \in W_*$ so that $\tau = wv$. If $|v| \geq k+1$, then $l(\tau_{[-1]}) = l(wv_{[-1]}) \leq \beta_2 l(w) \leq \beta_2 s$, which contradicts $\tau \in \Lambda_{\beta_2 s}$. Hence $|v| \leq k$. This shows that η is surjective, and $\#\Lambda_{\beta_2 s} \leq (\#S)^k \#\Lambda_s$ follows.

Let $\ell := \max\{j \in \mathbb{N}\cup\{0\} \mid s \leq \beta_2^j\}$. Then $\beta_2 < \beta_2^{-\ell} s \leq 1$. Therefore $\ell \leq (\log s)/\log\beta_2$ and $\#\Lambda_s \leq (\#S)^{k\ell} \#\Lambda_{\beta_2^{-\ell} s} \leq (\#S)^{(k\log s)/\log\beta_2} \#\Lambda_{\beta_2} = (\#\Lambda_{\beta_2}) s^{-\alpha}$. ■

Finally we define the notion of self-similar scales and prove a basic asymptotic property of these scales.

Definition 2.8 (Self-similar scales) Let $\boldsymbol{\alpha} = (\alpha_i)_{i\in S} \in (0,1)^S$. Define a gauge function $g_{\boldsymbol{\alpha}}$ on W_* by $g_{\boldsymbol{\alpha}}(w) := \alpha_w$, where $\alpha_{w_1\ldots w_m} := \alpha_{w_1}\ldots\alpha_{w_m}$ for $w_1\ldots w_m \in W_*$. Also let $\mathcal{S}(\boldsymbol{\alpha}) = \{\Lambda_s(\boldsymbol{\alpha})\}_{s\in(0,1]}$ be the scale induced by $g_{\boldsymbol{\alpha}}$. We call $\mathcal{S}(\boldsymbol{\alpha})$ the *self-similar scale with weight* $\boldsymbol{\alpha}$.

Clearly, any self-similar scale is elliptic.

Proposition 2.9 *Let $\boldsymbol{\alpha} = (\alpha_i)_{i\in S} \in (0,1)^S$ and let $d(\boldsymbol{\alpha})\ (\in [0,\infty))$ be the unique $d \in \mathbb{R}$ that satisfies $\sum_{i\in S} \alpha_i^d = 1$. Set $\alpha := \min_{i\in S} \alpha_i$. Then for any $s \in (0,1]$,*

$$s^{-d(\boldsymbol{\alpha})} \leq \#\Lambda_s(\boldsymbol{\alpha}) \leq \alpha^{-d(\boldsymbol{\alpha})} \cdot s^{-d(\boldsymbol{\alpha})}. \tag{2.2}$$

Proof. We will write Λ_s and d instead of $\Lambda_s(\boldsymbol{\alpha})$ and $d(\boldsymbol{\alpha})$ in this proof. Let μ be the Bernoulli measure on $\Sigma = S^{\mathbb{N}}$ with weight $(\alpha_i^d)_{i\in S}$. Let $s \in (0,1]$. By Theorem 2.5, $\alpha_{w_{[-1]}} > s \geq \alpha_w$, hence $\alpha s < \alpha_w \leq s$, for any $w \in \Lambda_s$. Since $\Sigma = \bigcup_{w\in\Lambda_s} \Sigma_w$ (disjoint),

$$(\alpha s)^d \#\Lambda_s = \sum_{w\in\Lambda_s} (\alpha s)^d \leq \sum_{w\in\Lambda_s} \alpha_w^d \Big(= \sum_{w\in\Lambda_s} \mu(\Sigma_w) = \mu(\Sigma) = 1\Big) \leq \sum_{w\in\Lambda_s} s^d = s^d \#\Lambda_s$$

and (2.2) is immediate from this. ■

2.2. *Self-similar structures and measures*

In this subsection we introduce the notion of self-similar structures and recall related definitions and results.

Definition 2.10 (Self-similar structures) (1) Let K be a compact metrizable space, S be a non-empty finite set and $F_i : K \to K$ be a continuous injection for each $i \in S$. The triple $(K, S, \{F_i\}_{i\in S})$ is called a *self-similar structure* if and only if there exists a continuous surjection $\pi : \Sigma = \Sigma(S) \to K$ such that $\pi\circ\sigma_i = F_i\circ\pi$ for each $i \in S$.
(2) Let $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ be a self-similar structure. For $w = w_1\ldots w_m \in W_*$, we set $F_w := F_{w_1}\circ\ldots\circ F_{w_m}$ and $K_w := F_w(K)$, where $F_\emptyset := id_K$ for $w = \emptyset$. We define the *critical set* $\mathcal{C}_\mathcal{L}$ and the *post critical set* $\mathcal{P}_\mathcal{L}$ of $\mathcal{L}$ by $\mathcal{C}_\mathcal{L} := \pi^{-1}\big(\cup_{i,j\in S, i\neq j}(K_i \cap K_j)\big)$ and $\mathcal{P}_\mathcal{L} := \cup_{m=1}^\infty \sigma^m(\mathcal{C}_\mathcal{L})$, respectively. We also set $V_0 := V_0(\mathcal{L}) := \pi(\mathcal{P}_\mathcal{L})$. Note that

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$\mathcal{P}_{\mathcal{L}} \in \mathcal{B}(\Sigma)$ and $V_0 \in \mathcal{B}(K)$.
(3) We say that $\mathcal{L}$ is *strongly finite* if and only if $\sup_{x\in K} \#(\pi^{-1}(x)) < \infty$, and that $\mathcal{L}$ is *post critically finite* (or simply *p.c.f.*) if and only if $\#\mathcal{P}_{\mathcal{L}} < \infty$.

Given a self-similar structure $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$, we always assume $\#K \geq 2$, and hence $\#S \geq 2$, to exclude the trivial case where K is just a one-point set. The set V_0 is regarded as the *'boundary'* of K. In fact, by [27, Proposition 1.3.5 (2)], if $w, v \in W_*$ and $\Sigma_w \cap \Sigma_v = \emptyset$ then $K_w \cap K_v = F_w(V_0) \cap F_v(V_0)$.

We fix a self-similar structure $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ in the rest of this subsection. The following easy lemma is fundamental for our study.

Lemma 2.11 *Assume $K \neq \overline{V_0}$. Set $K^I := K \setminus \overline{V_0}$ and $K^I_w := F_w(K^I)$ for each $w \in W_*$. Then K^I_w is an open subset of K and $K^I_w \subset K^I$ for any $w \in W_*$. Moreover, let Λ be a partition of Σ and set $K^I_\Lambda := \bigcup_{w\in\Lambda} K^I_w$. Then $K \setminus K^I_\Lambda = \bigcup_{w\in\Lambda} F_w(\overline{V_0})$.*

Proof. The first two statements follow from Kigami [28, Proof of Theorem 1.2.7], but we include the proof for ease of the reading. Let $w \in W_*$ and set $m := |w|$. Since $K \setminus K^I_w = F_w(\overline{V_0}) \cup \bigcup_{v\in W_m\setminus\{w\}} K_v \supset \bigcup_{v\in W_m} F_v(V_0) \supset V_0$, K^I_w is an open subset of K and $\overline{V_0} \subset K \setminus K^I_w$. Therefore $K^I_w \subset K \setminus \overline{V_0} = K^I$.

Next let $w \in \Lambda$. Then clearly $F_w(V_0) \subset K \setminus K^I_\Lambda$, hence $F_w(\overline{V_0}) = \overline{F_w(V_0)} \subset \overline{K \setminus K^I_\Lambda} = K \setminus K^I_\Lambda$ by the compactness of K. Therefore $\bigcup_{w\in\Lambda} F_w(\overline{V_0}) \subset K \setminus K^I_\Lambda$. The converse inclusion follows from $K = \bigcup_{w\in\Lambda} K_w = \bigcup_{w\in\Lambda}\bigl(K^I_w \cup F_w(\overline{V_0})\bigr) = K^I_\Lambda \cup \bigcup_{w\in\Lambda} F_w(\overline{V_0})$. ■

The following easy lemma is used (only) in Subsection 7.2.

Lemma 2.12 *Assume that $K \neq V_0$. Let Λ be a partition of Σ and $\Gamma \subset \Lambda$. Then for any $w \in \Lambda \setminus \Gamma$, $K_w \cap \mathrm{int}_K\bigl(\bigcup_{v\in\Gamma} K_v\bigr) = \emptyset$.*

Proof. Let $w \in \Lambda \setminus \Gamma$ and suppose $K_w \cap \mathrm{int}_K K(\Gamma) \neq \emptyset$. Then $U := F_w^{-1}\bigl(\mathrm{int}_K K(\Gamma)\bigr) = F_w^{-1}\bigl(K_w \cap \mathrm{int}_K K(\Gamma)\bigr)$ is a non-empty open subset of K. We have $U \subset V_0$ since $K_w \cap \mathrm{int}_K K(\Gamma) \subset \bigcup_{v\in\Gamma}(K_w \cap K_v) = \bigcup_{v\in\Gamma}(F_w(V_0) \cap F_v(V_0)) \subset F_w(V_0)$. Therefore $\mathrm{int}_K V_0 \neq \emptyset$, which contradicts $K \neq V_0$ by [27, Theorem 1.3.8]. Hence $K_w \cap \mathrm{int}_K K(\Gamma) = \emptyset$. ■

Next we consider some classes of Borel probability measures on K.

Definition 2.13 (1) We define a collection $\mathcal{M}(K)$ of Borel probability measures by

$$\begin{aligned}\mathcal{M}(K) := \{\mu \mid \mu &\text{ is a Borel probability measure on } K,\ \mu(\{x\}) = 0 \text{ for}\\ &\text{any } x \in K,\ \mu(K_w) > 0 \text{ and } \mu(F_w(V_0)) = 0 \text{ for any } w \in W_*\}.\end{aligned} \tag{2.3}$$

(2) A Borel probability measure μ on K is called *elliptic* if and only if the following holds:
(ELm) There exists $\gamma \in (0, \infty)$ such that $\mu(K_{wi}) \geq \gamma\mu(K_w)$ for any $(w, i) \in W_* \times S$.

By [28, Theorem 1.2.4], if $K \neq \overline{V_0}$ then every elliptic Borel probability measure on K belongs to $\mathcal{M}(K)$.

Definition 2.14 (Self-similar measures) Let $(\mu_i)_{i\in S} \in (0,1)^S$ satisfy $\sum_{i\in S} \mu_i = 1$. A Borel probability measure μ on K is called a *self-similar measure with weight* $(\mu_i)_{i\in S}$ if and only if the following equality (of Borel measures on K) holds:

$$\mu = \sum_{i\in S} \mu_i \mu \circ F_i^{-1}. \tag{2.4}$$

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Let $(\mu_i)_{i\in S} \in (0,1)^S$ satisfy $\sum_{i\in S} \mu_i = 1$. If ν is the Bernoulli measure on Σ with weight $(\mu_i)_{i\in S}$, then $\nu\circ\pi^{-1}$ is a self-similar measure on K with the same weight. Therefore there does exist a self-similar measure with the given weight. See [27, Section 1.4] for details.

Let μ be a self-similar measure with weight $(\mu_i)_{i\in S}$. If $K \neq \overline{V_0}$, then by [28, Theorem 1.2.7 and its proof], $\mu(K_w) = \mu_w$ and $\mu(F_w(\overline{V_0})) = 0$ for any $w \in W_*$. In particular, a self-similar measure with given weight is unique and elliptic in this case.

2.3. Systems of neighborhoods associated with scales

Let $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ be a self-similar structure. In this subsection, we define a fundamental system of neighborhoods $\{U_s^{(n)}(x, \mathcal{S})\}_{s\in(0,1]}$ of $x \in K$ associated with a scale $\mathcal{S} = \{\Lambda_s\}_{s\in(0,1]}$. Intuitively, $U_s^{(n)}(x, \mathcal{S})$ is a union of K_w's over $w \in \Lambda_s$ which are around x. $U_s^{(n)}(x, \mathcal{S})$ is regarded as a 'ball of radius s', although there may not be an associated distance. See [28, Chapter 2] for existence of such distances. We then introduce the notion of the *volume doubling property with respect to a scale* defined in [28, Section 1.3]. This property is closely related with (sub-)Gaussian heat kernel estimate, and will be mentioned again in Section 5.

In the rest of this subsection, we fix a self-similar structure $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ and a scale $\mathcal{S} = \{\Lambda_s\}_{s\in(0,1]}$ on Σ.

Definition 2.15 Let $\Gamma \subset W_*$ and $A \subset K$.
(1) We set $W(\Gamma, A) := \{w \in \Gamma \mid K_w \cap A \neq \emptyset\}$ and $K(\Gamma) := \bigcup_{w\in\Gamma} K_w$.
(2) Define $W^{(0)}(\Gamma, A) := W(\Gamma, A)$, and inductively, $K^{(n)}(\Gamma, A) := K(W^{(n)}(\Gamma, A))$ and $W^{(n+1)}(\Gamma, A) := W(\Gamma, K^{(n)}(\Gamma, A))$ for $n = 0, 1, 2, \dots$.

The following lemma is immediate by the above definitions.

Lemma 2.16 *Let $A \subset K$.*
(1) Let Λ be a partition of Σ. Then $A \subset \mathrm{int}_K(K^{(0)}(\Lambda, A))$, and for any $n \in \mathbb{N}\cup\{0\}$, $K^{(n)}(\Lambda, A) \subset \mathrm{int}_K(K^{(n+1)}(\Lambda, A))$ and $W^{(n)}(\Lambda, A) \subset W^{(n+1)}(\Lambda, A)$.
(2) Let Λ_i, $i = 1, 2$, be partitions of Σ with $\Lambda_1 \leq \Lambda_2$. Then for any $n \in \mathbb{N}\cup\{0\}$, $K^{(n)}(\Lambda_1, A) \subset K^{(n)}(\Lambda_2, A)$.

Definition 2.17 For $x \in K$, $s \in (0,1]$ and $n \in \mathbb{N}\cup\{0\}$, we define $\Lambda_{s,x}^n := W^{(n)}(\Lambda_s, \{x\})$ and $U_s^{(n)}(x, \mathcal{S}) := K^{(n)}(\Lambda_s, \{x\})$. We write $\Lambda_{s,x} := \Lambda_{s,x}^0$, $K_s(x, \mathcal{S}) := U_s^{(0)}(x, \mathcal{S})$ and $U_s(x, \mathcal{S}) := U_s^{(1)}(x, \mathcal{S})$. We also set $\Lambda_{s,w} := W(\Lambda_s, K_w)$ for $s \in (0,1]$ and $w \in W_*$.

Clearly, $\{U_s^{(n)}(x, \mathcal{S})\}_{s\in(0,1]}$ is decreasing as $s \downarrow 0$ and forms a fundamental system of neighborhoods of x in K.

Definition 2.18 (Locally finite scales) We say that $\mathcal{S}$ is *locally finite with respect to* $\mathcal{L}$, or simply $(\mathcal{L}, \mathcal{S})$ is *locally finite*, if and only if $\sup\{\#(\Lambda_{s,w}) \mid s \in (0,1], w \in \Lambda_s\} < \infty$.

Definition 2.19 (Volume doubling property) Let $\mu \in \mathcal{M}(K)$. For $n \in \mathbb{N}\cup\{0\}$, $(\mathcal{L}, \mathcal{S}, \mu)$ is said to satisfy $(\mathrm{VD})_n$ if and only if there exist $\alpha \in (0,1)$ and $c_V \in (0,\infty)$ such that $\mu(U_s^{(n)}(x)) \leq c_V \mu(U_{\alpha s}^{(n)}(x))$ for any $(s, x) \in (0,1]\times K$. We say that μ is *volume doubling with respect to* $\mathcal{S}$, or simply $(\mathcal{L}, \mathcal{S}, \mu)$ satisfies (VD), if and only if $(\mathcal{L}, \mathcal{S}, \mu)$ satisfies $(\mathrm{VD})_n$ for some $n \in \mathbb{N}$.

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2.4. Qdistances adapted to scales and cell-counting dimension

Next we introduce the notions of *qdistances* and *cell-counting dimension*. We continue to fix a self-similar structure $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ and a scale $\mathcal{S} = \{\Lambda_s\}_{s\in(0,1]}$ on Σ.

Definition 2.20 (Qdistances) Let E be a set, $\alpha \in (0,\infty)$ and $d : E \times E \to [0,\infty)$. Then d is said to be an *α-qdistance on E* if and only if $d^\alpha := d(\cdot,\cdot)^\alpha$ is a distance on E. Also d is called a *qdistance on E* if d is an α-qdistance for some $\alpha \in (0,\infty)$.

If d is an α-qdistance on E, then E is regarded as being equipped with the topology given by the distance d^α.

Notation. Let $d : K \times K \to [0,\infty)$. Then we set $B_r(x,d) := \{y \in K \mid d(x,y) < r\}$ for any $x \in K$ and any $r \in (0,\infty)$. We also set $\mathrm{diam}_d A := \sup_{y,z\in A} d(y,z)$ and $\mathrm{dist}_d(x,A) := \inf_{y\in A} d(x,y)$ for any $x \in K$ and any non-empty $A \subset K$.

Definition 2.21 A qdistance d on K is said to be *adapted to* $\mathcal{S}$ if and only if there exist $\beta_1, \beta_2 \in (0,\infty)$ and $n \in \mathbb{N}$ such that for any $(s,x) \in (0,1] \times K$,

$$B_{\beta_1 s}(x,d) \subset U_s^{(n)}(x,\mathcal{S}) \subset B_{\beta_2 s}(x,d). \tag{2.5}$$

If d is adapted to $\mathcal{S}$, then $\{U_s^{(n)}(x,\mathcal{S})\}_{s\in(0,1],x\in K}$ may be thought of as real balls. Since $\{U_s^{(n)}(x,\mathcal{S})\}_{s\in(0,1]}$ is a fundamental system of neighborhoods of x, the topology determined by d is the same as the original one of K in this case.

Lemma 2.22 *Let $\mu \in \mathcal{M}(K)$, let d be a qdistance on K adapted to $\mathcal{S}$ and let $n \in \mathbb{N}$ be as in Definition 2.21. Then $(\mathcal{L}, \mathcal{S}, \mu)$ satisfies* $(\mathrm{VD})_n$ *if and only if there exists $c_V \in (0,\infty)$ such that for any $(r,x) \in (0,\infty) \times K$,*

$$\mu(B_{2r}(x,d)) \le c_V \mu(B_r(x,d)). \tag{2.6}$$

Proof. Note that $\inf_{x\in K} \mu(B_r(x,d)) > 0$ for a fixed $r \in (0,\infty)$, since $x \mapsto \mu(B_r(x,d))$ is a $(0,\infty)$-valued lower semicontinuous function on a compact space K. Now the statement is straightforward from (2.5). ■

Definition 2.23 (Cell-counting dimension) Let $\eta \in [0,\infty)$ and $A \subset K$. We say that the *cell-counting dimension of A with respect to $\mathcal{S}$ is bounded from above* (resp. *below*) *by* η, and write $\dim_{\mathcal{S}} A \le \eta$ (resp. $\dim_{\mathcal{S}} A \ge \eta$), if and only if $\sup_{s\in(0,1]} s^\eta \#W(\Lambda_s, A) < \infty$ (resp. $\inf_{s\in(0,1]} s^\eta \#W(\Lambda_s, A) > 0$). We call η the *cell-counting dimension of A with respect to* $\mathcal{S}$, and write $\dim_{\mathcal{S}} A = \eta$, if and only if both $\dim_{\mathcal{S}} A \le \eta$ and $\dim_{\mathcal{S}} A \ge \eta$ hold. Note that $\eta \in [0,\infty)$ satisfying $\dim_{\mathcal{S}} A = \eta$, if exists, is unique.

The notion of cell-counting dimension corresponds to that of box-counting dimension in the settings of metric spaces. In fact, we have the following proposition.

Proposition 2.24 *Let d be a qdistance on K adapted to $\mathcal{S}$, let $A \subset K$ and $\eta \in [0,\infty)$. For $r \in (0,\infty)$, let $\mathcal{N}_r(A)$ be the smallest number N of balls $\{B_r(x_i,d)\}_{i=1}^N$ of radius r that can cover A. Suppose that $(\mathcal{L}, \mathcal{S})$ is locally finite. Then $\dim_{\mathcal{S}} A \le \eta$ (resp. $\dim_{\mathcal{S}} A \ge \eta$) if and only if $\sup_{r\in(0,1]} r^\eta \mathcal{N}_r(A) < \infty$ (resp. $\inf_{r\in(0,1]} r^\eta \mathcal{N}_r(A) > 0$).*

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Proof. Take $\beta_1, \beta_2 > 0$ and $n \in \mathbb{N}$ so that (2.5) holds. We may assume that $\beta_1 \leq 1 \leq \beta_2$.
Let $s \in (0,1]$. We choose $x_w \in K_w$ for each $w \in W(\Lambda_s, A)$. Then

$$A \subset \bigcup_{w \in W(\Lambda_s, A)} K_w \subset \bigcup_{w \in W(\Lambda_s, A)} U_s^{(n)}(x_w, \mathcal{S}) \subset \bigcup_{w \in W(\Lambda_s, A)} B_{\beta_2 s}(x_w, d),$$

so $\mathcal{N}_{\beta_2 s}(A) \leq \# W(\Lambda_s, A)$. Therefore

$$\beta_2^{-\eta} \inf_{r \in (0, \beta_2]} r^\eta \mathcal{N}_r(A) \leq \inf_{s \in (0,1]} s^\eta \# W(\Lambda_s, A), \tag{2.7}$$

$$\sup_{r \in (0, \beta_2]} r^\eta \mathcal{N}_r(A) \leq \beta_2^\eta \sup_{s \in (0,1]} s^\eta \# W(\Lambda_s, A). \tag{2.8}$$

By (2.8), $\dim_{\mathcal{S}} A \leq \eta$ implies $\sup_{r \in (0,1]} r^\eta \mathcal{N}_r(A) < \infty$. Suppose $\inf_{r \in (0,1]} r^\eta \mathcal{N}_r(A) > 0$. Then $A \neq \emptyset$ and $\mathcal{N}_r(A) \geq 1$ for any $r > 0$. Therefore $\inf_{r \in [1, \beta_2]} r^\eta \mathcal{N}_r(A) \geq 1$ and $\inf_{r \in (0, \beta_2]} r^\eta \mathcal{N}_r(A) > 0$. Now this and (2.7) implies $\dim_{\mathcal{S}} A \geq \eta$.

For the converse implications, let $M := \sup\{\# \Lambda_{s,x}^{n+1} \mid s \in (0,1], x \in K\}$. Since $(\mathcal{L}, \mathcal{S})$ is locally finite, $M < \infty$ by [28, Lemma 1.3.6]. Let $s \in (0,1]$ and $N := \mathcal{N}_{\beta_1 s}(A)$ and choose $\{x_i\}_{i=1}^N \subset K$ so that $A \subset \bigcup_{i=1}^N B_{\beta_1 s}(x_i, d) \big(\subset \bigcup_{i=1}^N U_s^{(n)}(x_i, \mathcal{S}) \big)$. If $w \in W(\Lambda_s, A)$, $U_s^{(n)}(x_i, \mathcal{S}) \cap K_w \neq \emptyset$, hence $w \in \Lambda_{s, x_i}^{n+1}$ for some $i \in \{1, \ldots, N\}$. Therefore $W(\Lambda_s, A) \subset \bigcup_{i=1}^N \Lambda_{s, x_i}^{n+1}$ and $\# W(\Lambda_s, A) \leq \sum_{i=1}^N \# \Lambda_{s, x_i}^{n+1} \leq MN = M \mathcal{N}_{\beta_1 s}(A)$. This yields

$$M^{-1} \beta_1^\eta \inf_{s \in (0,1]} s^\eta \# W(\Lambda_s, A) \leq \inf_{r \in (0, \beta_1]} r^\eta \mathcal{N}_r(A), \tag{2.9}$$

$$\sup_{s \in (0,1]} s^\eta \# W(\Lambda_s, A) \leq M \beta_1^{-\eta} \sup_{r \in (0, \beta_1]} r^\eta \mathcal{N}_r(A). \tag{2.10}$$

If $\dim_{\mathcal{S}} A \geq \eta$, then $A \neq \emptyset$ and $\inf_{r \in [\beta_1, 1]} r^\eta \mathcal{N}_r(A) \geq \beta_1^\eta$, which together with (2.9) implies $\inf_{r \in (0,1]} r^\eta \mathcal{N}_r(A) > 0$. On the other hand, by (2.10), $\sup_{r \in (0,1]} r^\eta \mathcal{N}_r(A) < \infty$ implies $\dim_{\mathcal{S}} A \leq \eta$. This completes the proof. ■

3. Framework: Self-similar Dirichlet spaces

In this section, we introduce our framework of spectral analysis on self-similar structures, which we call *self-similar Dirihlet spaces*. See Fukushima, Oshima and Takeda [17] for basic notions concerning Dirichlet forms on locally compact separable metrizable spaces.

The following lemma is immediate from the results of Subsection 2.2.

Lemma 3.1 *Let $(K, S, \{F_i\}_{i \in S})$ be a self-similar structure, $\mu \in \mathcal{M}(K)$ and $w \in W_*$.*
(1) The Borel probability measure μ^w on K defined by $\mu^w := \mu(K_w)^{-1} \mu \circ F_w$ belongs to $\mathcal{M}(K)$, and $\int_K u \circ F_w d\mu^w = \mu(K_w)^{-1} \int_{K_w} u d\mu$ for any $u : K \to [0, \infty]$ Borel measurable. In particular, if we set $\rho_w u := u \circ F_w$ for $u : K \to [-\infty, \infty]$, then ρ_w defines a bounded linear operator $\rho_w : L^2(K, \mu) \to L^2(K, \mu^w)$.
(2) If μ is a self-similar measure and $K \neq \overline{V_0}$, then $\mu^w = \mu$.

Definition 3.2 For $u : K \to \mathbb{R}$, $w \in W_*$, define $u^w : K \to \mathbb{R}$ by $u^w := \begin{cases} u \circ F_w^{-1} & \text{on} \quad K_w \\ 0 & \text{on} \quad K \setminus K_w. \end{cases}$
Clearly, if u is Borel measurable then so is u^w for any $w \in W_*$.

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Now we introduce the notion of self-similar Dirichlet spaces. Note that under the situation of the next definition, we can regard $\mathcal{F}\cap C(K)$ as a subspace of $C(K)$, hence $u\circ F_i\,(\in C(K))$ as an element of $L^2(K,\mu)$ for $u\in\mathcal{F}\cap C(K)$.

Definition 3.3 (Self-similar Dirichlet spaces) Let $\mathcal{L}=(K,S,\{F_i\}_{i\in S})$ be a self-similar structure satisfying $K\neq\overline{V_0}$ and let μ be an elliptic Borel probability measure on K. A (symmetric) regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(K,\mu)$ is called *self-similar with resistance scaling ratio* $\mathbf{r}=(r_i)_{i\in S}\in(0,\infty)^S$ if and only if the following four conditions are satisfied:
(SSDF1) $u\circ F_i\in\mathcal{F}\cap C(K)$ for any $u\in\mathcal{F}\cap C(K)$ and any $i\in S$.
(SSDF2) For any $u,v\in\mathcal{F}\cap C(K)$,

$$\mathcal{E}(u,v)=\sum_{i\in S}\frac{1}{r_i}\mathcal{E}(u\circ F_i,v\circ F_i). \tag{3.1}$$

(SSDF3) $u^i\in\mathcal{F}\cap C(K)$ for any $i\in S$ and any $u\in\mathcal{F}\cap C(K)$ with $\mathrm{supp}_K[u]\subset K^I(:=K\setminus\overline{V_0}$, recall Lemma 2.11), where u^i is as in Definition 3.2.
(SSDF4) The function $g:W_*\to(0,\infty)$ defined by $g(w):=\sqrt{r_w\mu(K_w)}$ is a gauge function on W_* and the scale induced by g is elliptic.

If $(\mathcal{E},\mathcal{F})$ is a self-similar regular Dirichlet form on $L^2(K,\mu)$ with resistance scaling ratio $\mathbf{r}=(r_i)_{i\in S}$, then we call $(\mathcal{L},\mu,\mathcal{E},\mathcal{F},\mathbf{r})$ a *self-similar Dirichlet space.*

Remark. (1) For a self-similar Dirichlet space $(\mathcal{L}=(K,S,\{F_i\}_{i\in S}),\mu,\mathcal{E},\mathcal{F},\mathbf{r}=(r_i)_{i\in S})$,
(i) $\mu\in\mathcal{M}(K)$ (by [28, Theorem 1.2.4]).
(ii) $\boldsymbol{1}\in\mathcal{F}$ (by the compactness of K and the regularity of $(\mathcal{E},\mathcal{F})$).
(2) If μ is a self-similar measure with weight $(\mu_i)_{i\in S}$, then (SSDF4) is equivalent to the condition that $r_i\mu_i<1$ for any $i\in S$.

In the rest of this section, $(\mathcal{L}=(K,S,\{F_i\}_{i\in S}),\mu,\mathcal{E},\mathcal{F},\mathbf{r}=(r_i)_{i\in S})$ is assumed to be a self-similar Dirichlet space.

Notation. Set $g(w):=\sqrt{r_w\mu(K_w)}$ for $w\in W_*$ and let $\mathcal{S}=\{\Lambda_s\}_{s\in(0,1]}$ be the scale on Σ induced by the gauge function g. We write $\mathcal{E}_1(u,v):=\mathcal{E}(u,v)+\int_K uv\,d\mu$ for $u,v\in\mathcal{F}$. Also for $A\in\mathcal{B}(K)$, we write $\mu|_A:=\mu|_{\mathcal{B}(A)}$.

We state several preliminary results on $(\mathcal{L},\mu,\mathcal{E},\mathcal{F},\mathbf{r})$ needed in the following sections.

Lemma 3.4 *$(\mathcal{E},\mathcal{F})$ is a local Dirichlet form.*

Proof. Let $u,v\in\mathcal{F}\cap C(K)$ with $u,v\neq 0$ and $\mathrm{supp}_K[u]\cap\mathrm{supp}_K[v]=\emptyset$. Since $\mathrm{supp}_K[u]$ and $\mathrm{supp}_K[v]$ are compact, we can choose $m\in\mathbb{N}$ so that for each $w\in W_m$, either $K_w\cap\mathrm{supp}_K[u]=\emptyset$ or $K_w\cap\mathrm{supp}_K[v]=\emptyset$ holds. Then by (3.1) we have

$$\mathcal{E}(u,v)=\sum_{w\in W_m}\frac{1}{r_w}\mathcal{E}(u\circ F_w,v\circ F_w)=0.$$

Now the local property of $(\mathcal{E},\mathcal{F})$ follows by [17, Problem 1.4.1 and Theorem 3.1.2]. ■

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Definition 3.5 Let U be a non-empty open subset of K. Define

$$\mathcal{C}_U := \{u \in \mathcal{F} \cap C(K) \mid \operatorname{supp}_K[u] \subset U\} \quad \text{and} \quad \mathcal{F}_U := \overline{\mathcal{C}_U}, \tag{3.2}$$

where the closure is taken in the Hilbert space $(\mathcal{F}, \mathcal{E}_1)$. We also set $\mathcal{E}^U := \mathcal{E}|_{\mathcal{F}_U \times \mathcal{F}_U}$. We call $(\mathcal{E}^U, \mathcal{F}_U)$ the *part of the Dirichlet form* $(\mathcal{E}, \mathcal{F})$ *on* U.

Since $u = 0$ μ-a.e. on $K \setminus U$ for any $u \in \mathcal{F}_U$, we can regard $\mathcal{F}_U$ as a subspace of $L^2(U, \mu|_U)$ in the natural way. Then by [17, Theorem 1.4.2 (v) and Lemma 1.4.2 (ii)], we easily see that $(\mathcal{E}^U, \mathcal{F}_U)$ is a local regular Dirichlet form on $L^2(U, \mu|_U)$.

Lemma 3.6 *Let $w \in W_*$. Then $u^w \in \mathcal{C}_{K^I_w}$ for any $u \in \mathcal{C}_{K^I}$ and $\rho_w(\mathcal{C}_{K^I_w}) = \mathcal{C}_{K^I}$.*

Proof. $\rho_w(\mathcal{C}_{K^I_w}) \subset \mathcal{C}_{K^I}$ is clear by (SSDF1). Conversely if $u \in \mathcal{C}_{K^I}$, then using (SSDF3) repeatedly, we have $u^w \in \mathcal{C}_{K^I_w}$. Hence $u = u^w \circ F_w \in \rho_w(\mathcal{C}_{K^I_w})$. ■

The following lemma is used (only) in Subsection 7.2.

Lemma 3.7 *There exist $c, \alpha \in (0, \infty)$ such that $c\mu(K_w) \geq s^\alpha$ for any $s \in (0, 1]$, $w \in \Lambda_s$.*

Proof. Since the scale $\mathcal{S} = \{\Lambda_s\}_{s \in (0,1]}$ is assumed to be elliptic by (SSDF4), we easily see that there exists $\beta_1 \in (0, 1)$ such that $g(w) \geq \beta_1 s$ for any $s \in (0, 1]$ and any $w \in \Lambda_s$. It is also easy to show that there exist $c_1 \in (0, \infty)$ and $\beta_2 \in (0, 1)$ such that $g(w) \leq c_1 \beta_2^{|w|}$ for any $w \in W_*$. Since μ is also assumed to be elliptic, we can choose $\gamma \in (0, 1)$ so that $\mu(K_{wi}) \geq \gamma\mu(K_w)$ for any $w \in W_*$ and any $i \in S$. Then $\mu(K_w) \geq \gamma^{|w|}$ for any $w \in W_*$. Now set $\alpha := (\log \gamma)/\log \beta_2 \, (\in (0, \infty))$ and let $s \in (0, 1]$ and $w \in \Lambda_s$. Then

$$\beta_1 s \leq g(w) \leq c_1 \beta_2^{|w|} = c_1 \gamma^{|w|/\alpha} \leq c_1 \mu(K_w)^{1/\alpha}.$$

Thus $(c_1/\beta_1)^\alpha \mu(K_w) \geq s^\alpha$. ■

4. Spectral and geometric counting functions

Now we start to study spectral properties of self-similar Dirichlet forms. In this section, we state and prove our first main result (Theorem 4.3). Throughout this section, let $(\mathcal{L} = (K, S, \{F_i\}_{i \in S}), \mu, \mathcal{E}, \mathcal{F}, \mathbf{r} = (r_i)_{i \in S})$ be a self-similar Dirichlet space and $\mathcal{S} = \{\Lambda_s\}_{s \in (0,1]}$ be the scale induced by the gauge function $g : w \mapsto \sqrt{r_w \mu(K_w)}$.

First we define the eigenvalue counting and partition functions of a non-negative self-adjoint operator on a Hilbert space. Note that, in the present setting, $L^2(U, \mu|_U)$ is an infinite-dimensional separable Hilbert space for any $U \subset K$ non-empty open.

Definition 4.1 (Eigenvalue counting and partition functions) Let H be a non-negative self-adjoint operator on an infinite-dimensional separable Hilbert space $\mathcal{H}$.
(1) The *partition function* Z_H of H (or of the contraction semigroup $\{e^{-tH}\}_{t \in (0,\infty)}$ or of the corresponding closed form on $\mathcal{H}$) is defined by $Z_H(t) := \operatorname{Tr}(e^{-tH})$, $t \in (0, \infty)$.
(2) Suppose that H has compact resolvent and let $\{\lambda^H_n\}_{n \in \mathbb{N}}$ be the non-decreasing enumeration of the eigenvalues of H, where each eigenvalue is repeated according to its multiplicity. The *eigenvalue counting function* N_H of H is defined by

$$N_H(x) := \#(\{n \in \mathbb{N} \mid \lambda^H_n \leq x\}), \qquad x \in [0, \infty), \tag{4.1}$$

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and then we have the following equalities for Z_H:

$$Z_H(t) = \mathrm{Tr}(e^{-tH}) = \sum_{n\in\mathbb{N}} e^{-t\lambda_n^H} = \int_{[0,\infty)} e^{-ts} dN_H(s), \qquad t \in (0,\infty). \tag{4.2}$$

Note that $N_H(x) < \infty$ for any $x \in [0,\infty)$ since $\lim_{n\to\infty} \lambda_n^H = \infty$, and that Z_H is $(0,\infty)$-valued, strictly decreasing and continuous provided $Z_H(t) < \infty$ for any $t \in (0,\infty)$.

Notation. Let H_N (resp. H_D) be the non-negative self-adjoint operator associated with the closed form $(\mathcal{E},\mathcal{F})$ on $L^2(K,\mu)$ (resp. $(\mathcal{E}^{K^I},\mathcal{F}_{K^I})$ on $L^2(K^I,\mu|_{K^I})$). For $b \in \{N,D\}$, if H_b has compact resolvent, then we write $\lambda_n^b := \lambda_n^{H_b}$ and $N_b := N_{H_b}$.

Definition 4.2 (Uniform Poincaré inequality) We say that $(\mathcal{E},\mathcal{F})$ satisfies *Uniform Poincaré inequality*, (PI) for short, if and only if there exists $C_{\mathrm{PI}} \in (0,\infty)$ such that

$$\mathcal{E}(u,u) \geq C_{\mathrm{PI}} \int_K \left|u - \overline{u}^{\mu^w}\right|^2 d\mu^w, \qquad u \in \rho_w(\mathcal{F}\cap C(K)) \tag{PI}$$

for any $w \in W_*$, where $\overline{u}^{\nu} := \int_K u d\nu$ for a Borel probability measure ν on K.

Uniform Poincaré inequality yields the following estimate for the eigenvalue counting functions N_N and N_D, which is the main theorem of this section.

Theorem 4.3 *Assume that $(\mathcal{E},\mathcal{F})$ is conservative, i.e. $\mathcal{E}(\mathbf{1},\mathbf{1}) = 0$, and satisfies* (PI). *Then there exist $c_1, c_2 \in (0,\infty)$ and $\delta \in [1,\infty)$ such that for any $x \in [\delta,\infty)$,*

$$c_1 \#\Lambda_{x^{-1/2}} \leq N_D(x) \leq N_N(x) \leq c_2 \#\Lambda_{x^{-1/2}}. \tag{4.3}$$

Remark. In the arguments below, we will prove that H_N and H_D have compact resolvents under the situation of Theorem 4.3.

We provide a few simple sufficient conditions for (PI) before proving Theorem 4.3.

Proposition 4.4 (PI) *holds for each of the following two cases.*
(1) $\mathcal{F} \subset C(K)$, $(\mathcal{E},\mathcal{F})$ is a resistance form on K and its associated resistance metric R is compatible with the original topology of K.
(2) μ is a self-similar measure and there exists $C_{\mathrm{PI}} \in (0,\infty)$ such that

$$\mathcal{E}(u,u) \geq C_{\mathrm{PI}} \int_K \left|u - \overline{u}^{\mu}\right|^2 d\mu, \qquad u \in \mathcal{F}\cap C(K). \tag{4.4}$$

Proof. (1) This is immediate by [28, Proof of Lemma B.2]. (See Kigami [27, Chapter 2] and [29, Part I] for the definition and basic properties of resistance forms.)
(2) trivially yields (PI) since $\rho_w(\mathcal{F}\cap C(K)) \subset \mathcal{F}\cap C(K)$ and $\mu^w = \mu$ for any $w \in W_*$. ■

The rest of this section is devoted to the proof of Theorem 4.3. The proof is split into several lemmas and is based on the so-called *minimax principle* or the *variational formula* for the eigenvalues of non-negative self-adjoint operators. See Davies [15, Chapter 4] for details about the minimax principle. We first show the upper inequality of (4.3).

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Lemma 4.5 *Suppose that* $(\mathcal{E},\mathcal{F})$ *is conservative and satisfies* (PI). *Define*

$$\lambda(L) := \sup\Big\{\mathcal{E}(u,u)\,\Big|\, u\in L, \int_K |u|^2 d\mu = 1\Big\}, \qquad L\subset \mathcal{F}\cap C(K) \text{ subspace}, \tag{4.5}$$

$$\lambda_n := \inf\{\lambda(L) \mid L \text{ is an } n\text{-dimensional subspace of } \mathcal{F}\cap C(K)\}. \tag{4.6}$$

Let Λ *be a partition of* Σ. *Then*

$$\lambda_{\#\Lambda+1} \geq C_{\mathrm{PI}}\Big(\max_{w\in\Lambda} r_w\mu(K_w)\Big)^{-1}. \tag{4.7}$$

In particular, H_N *has compact resolvent, so does* H_D *and* $\lambda_n = \lambda_n^N$ *for any* $n\in\mathbb{N}$.

Proof. The statements of the final sentence follows from (4.7) in view of the minimax principle, $\mathcal{C}_{K^I}\subset\mathcal{F}\cap C(K)$ and (SSDF4), so it suffices to show (4.7). Note that we may regard $\rho_w(\mathcal{F}\cap C(K))$ as a subspace of $L^2(K,\mu^w)$ for $w\in W_*$. Also, regarded as subspaces of $C(K)$, $\rho_w(\mathcal{F}\cap C(K))\subset\mathcal{F}\cap C(K)$ by (SSDF1). Under these identifications, we define

$$\begin{aligned} \mathcal{F}_{N,\Lambda} &:= \{u\in L^2(K,\mu) \mid u\circ F_w \in \rho_w(\mathcal{F}\cap C(K)) \text{ for any } w\in\Lambda\},\\ \mathcal{E}^{N,\Lambda}(u,v) &:= \sum_{w\in\Lambda}\frac{1}{r_w}\mathcal{E}(u\circ F_w, v\circ F_w), \qquad u,v\in\mathcal{F}_{N,\Lambda}. \end{aligned} \tag{4.8}$$

Similarly to (4.5) and (4.6), we set

$$\lambda(L) := \sup\Big\{\mathcal{E}^{N,\Lambda}(u,u)\,\Big|\, u\in L, \int_K |u|^2 d\mu = 1\Big\}, \qquad L\subset \mathcal{F}_{N,\Lambda} \text{ subspace}, \tag{4.9}$$

$$\lambda_n^\Lambda := \inf\{\lambda(L) \mid L \text{ is an } n\text{-dimensional subspace of } \mathcal{F}_{N,\Lambda}\}. \tag{4.10}$$

$\mathcal{F}\cap C(K)\subset\mathcal{F}_{N,\Lambda}$ by definition, and $\mathcal{E}^{N,\Lambda}$ coincides with $\mathcal{E}$ on $(\mathcal{F}\cap C(K))\times(\mathcal{F}\cap C(K))$ by (SSDF2). Hence $\lambda_n\geq\lambda_n^\Lambda$ for any $n\in\mathbb{N}$.

Let $L_0 := \{\sum_{w\in\Lambda} a_w \boldsymbol{1}_{K_w} \mid a_w\in\mathbb{R} \text{ for each } w\in\Lambda\}$. Note that L_0 is a $\#\Lambda$-dimensional subspace of $\mathcal{F}_{N,\Lambda}$ and $\mathcal{E}^{N,\Lambda}|_{L_0\times L_0}\equiv 0$. Let $L\subset\mathcal{F}_{N,\Lambda}$ be a $(\#\Lambda+1)$-dimensional subspace and set $\widetilde{L} := L + L_0$. Then the bilinear form $\mathcal{E}^{N,\Lambda}$ on $\widetilde{L}$ is naturally associated with a non-negative self-adjoint operator A on $\widetilde{L}$ by the equality $\mathcal{E}^{N,\Lambda}(u,v) = \int_K Au\cdot v d\mu$, $u,v\in\widetilde{L}$. By the theory of finite-dimensional real symmetric matrices, the $(\#\Lambda+1)$-th smallest eigenvalue λ_A of A is given by

$$\lambda_A = \inf\big\{\lambda(L') \mid L' \text{ is a } (\#\Lambda+1)\text{-dimensional subspace of } \widetilde{L}\big\},$$

where $\lambda(L')$ is as in (4.9). Moreover, the eigenfunction $u\in\widetilde{L}$ corresponding to λ_A is orthogonal to L_0, that is, $\int_K u\circ F_w d\mu^w = \mu(K_w)^{-1}\int_{K_w} u d\mu = 0$ for any $w\in\Lambda$. We can normalize u so that $\int_K |u|^2 d\mu = 1$. Then by (PI),

$$\begin{aligned} \lambda(L) \geq \lambda_A = \mathcal{E}^{N,\Lambda}(u,u) &= \sum_{w\in\Lambda}\frac{1}{r_w}\mathcal{E}(u\circ F_w, u\circ F_w) \geq C_{\mathrm{PI}}\sum_{w\in\Lambda}\frac{1}{r_w}\int_K |u\circ F_w|^2 d\mu^w\\ &= C_{\mathrm{PI}}\sum_{w\in\Lambda}\frac{1}{r_w\mu(K_w)}\int_{K_w} u^2 d\mu \geq \frac{C_{\mathrm{PI}}}{\max_{w\in\Lambda} r_w\mu(K_w)}. \end{aligned}$$

Taking the infimum over L yields (4.7). ■

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Lemma 4.6 *Assume that $(\mathcal{E},\mathcal{F})$ is conservative and satisfies (PI). Then there exists $c_2 \in (0,\infty)$ such that for any $x \in [1,\infty)$,*

$$N_N(x) \le c_2 \#\Lambda_{x^{-1/2}}. \tag{4.11}$$

Proof. Let $s \in (0,1]$. By (4.7), $\lambda^N_{\#\Lambda_s+1} \ge C_{\mathrm{PI}}\bigl(\max_{w\in\Lambda_s} r_w\mu(K_w)\bigr)^{-1} \ge C_{\mathrm{PI}} s^{-2}$, hence $N_N(C_{\mathrm{PI}}s^{-2}/2) \le \#\Lambda_s$. We may assume $1 \ge C_{\mathrm{PI}}/2\,(=: \alpha)$. Let $x \in [1,\infty)$ and set $s^2 := \alpha/x\,(\in (0,1])$. Then $N_N(x) \le \#\Lambda_{\sqrt{\alpha}x^{-1/2}}$. Proposition 2.7 implies that there exists $c_2 > 0$ such that $\#\Lambda_{\sqrt{\alpha}t} \le c_2\#\Lambda_t$ for any $t \in (0,1]$. Thus the result follows. ■

Next we prove the lower bound of (4.3).

Lemma 4.7 *There exists $C_D \in (0,\infty)$ such that for any $w \in W_*$,*

$$\lambda_1(K^I_w) := \inf_{u\in\mathcal{C}_{K^I_w}, u\not\equiv 0} \frac{\mathcal{E}(u,u)}{\int_{K^I_w}|u|^2 d\mu} \le \frac{C_D}{r_w\mu(K_w)}. \tag{4.12}$$

Proof. Take $v \in W_*$ so that $K_v \subset K^I$. By the regularity of $(\mathcal{E},\mathcal{F})$ and [17, Problem 1.4.1], there exists $u \in \mathcal{C}_{K^I}$ such that $u \ge 0$ on K and $u = 1$ on K_v. Let $w \in W_*$. Then Lemma 3.6 implies that $u^w \in \mathcal{C}_{K^I_w}$. By (SSDF2) and the ellipticity of μ,

$$\lambda_1(K^I_w) \le \frac{\mathcal{E}(u^w,u^w)}{\int_K |u^w|^2 d\mu} = \frac{\mathcal{E}(u,u)}{r_w\int_K |u^w|^2 d\mu} \le \frac{\mathcal{E}(u,u)}{r_w\mu(K_{wv})} \le \frac{\mathcal{E}(u,u)}{\gamma^{|v|}}\frac{1}{r_w\mu(K_w)},$$

where γ is the constant given in (ELm) (Definition 2.13 (2)). Since $u \in \mathcal{C}_{K^I}$ and $v \in W_*$ is independent of $w \in W_*$, (4.12) has been proved. ■

Lemma 4.8 *Assume that H_D has compact resolvent. For each $w \in W_*$, let H_w be the non-negative self-adjoint operator on $L^2(K^I_w, \mu|_{K^I_w})$ associated with $(\mathcal{E}^{K^I_w}, \mathcal{F}_{K^I_w})$. Let Λ be a partition of Σ and let H_Λ be the non-negative self-adjoint operator on $L^2(K^I_\Lambda, \mu|_{K^I_\Lambda})$ associated with $(\mathcal{E}^{K^I_\Lambda}, \mathcal{F}_{K^I_\Lambda})$ (recall Lemma 2.11). Then H_w and H_Λ have compact resolvents. Moreover, if we set $N_{K^I_w} := N_{H_w}$ and $N_{K^I_\Lambda} := N_{H_\Lambda}$, then for any $x \in [0,\infty)$,*

$$\sum_{w\in\Lambda} N_{K^I_w}(x) = N_{K^I_\Lambda}(x) \le N_D(x). \tag{4.13}$$

Proof. If $w \in \Lambda$, then by $\mathcal{F}_{K^I_w} \subset \mathcal{F}_{K^I_\Lambda} \subset \mathcal{F}_{K^I}$ and the minimax principle, H_w and H_Λ have compact resolvents and the inequality in (4.13) holds. So we show the equality in (4.13). The self-similarity of $(\mathcal{E},\mathcal{F})$ implies that $\mathcal{E}(u_1,u_2) = 0$ for any $w_i \in \Lambda$, $i = 1,2$ with $w_1 \ne w_2$ and any $u_i \in \mathcal{F}_{K^I_{w_i}}$, $i = 1,2$.

Let $w \in \Lambda$ and $u \in \mathcal{F}_{K^I_\Lambda}$. Since $K \setminus K^I_w = F_w(\overline{V_0}) \cup \bigcup_{\tau\in\Lambda\setminus\{w\}} K_\tau$, $(L_w :=) K_w \cap \mathrm{supp}_K[u] \subset K^I_w$. Therefore $u \cdot \boldsymbol{1}_{K^I_w} \in C(K)$ and $\mathrm{supp}_K[u\cdot\boldsymbol{1}_{K^I_w}] \subset L_w \subset K^I_w$. Since L_w is compact and K^I_w is open in K, we may take $\varphi_w \in \mathcal{F}\cap C(K)$ such that $\varphi_w \ge 0$, $\varphi_w|_{L_w} = 1$ and $\varphi_w|_{K\setminus K^I_w} = 0$ by [17, Problem 1.4.1]. Then $u\cdot\boldsymbol{1}_{K^I_w} = u\cdot\varphi_w \in \mathcal{F}$ by [17, Theorem 1.4.2. (ii)], hence $u\cdot\boldsymbol{1}_{K^I_w} \in \mathcal{C}_{K^I_w}$. It follows that $\mathcal{C}_{K^I_\Lambda} = \bigoplus_{w\in\Lambda}\mathcal{C}_{K^I_w}$, where $\mathcal{C}_{K^I_w}$, $w \in \Lambda$ are orthogonal to each other with respect to both $\mathcal{E}$ and the inner product of $L^2(K,\mu)$. Therefore taking the closure of both sides in the Hilbert space $(\mathcal{F},\mathcal{E}_1)$ leads

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to the equality $\mathcal{F}_{K^I_\Lambda} = \bigoplus_{w\in\Lambda} \mathcal{F}_{K^I_w}$ and again $\mathcal{F}_{K^I_w}$, $w \in \Lambda$ are orthogonal to each other with respect to both $\mathcal{E}$ and the inner product of $L^2(K,\mu)$. This fact immediately implies that each eigenspace of H_Λ is the direct sum over $w \in \Lambda$ of those of H_w with the same eigenvalue. Now the desired equality is obvious. ■

Lemma 4.9 *Suppose that H_D has compact resolvent. Then there exist $c_1 \in (0,\infty)$ and $\delta \in [1,\infty)$ such that for any $x \in [\delta,\infty)$,*

$$c_1 \#\Lambda_{x^{-1/2}} \leq N_D(x). \tag{4.14}$$

Proof. Since the gauge function g of $\mathcal{S} = \{\Lambda_s\}_{s\in(0,1]}$ is assumed to satisfy (EL1), we may choose $\beta \in (0,1)$ so that $g(w) \geq \beta s$ for any $s \in (0,1]$ and any $w \in \Lambda_s$. Let $s \in (0,1]$. Then by Lemma 4.7, we have

$$\lambda_1(K^I_w) \leq \frac{C_D}{r_w \mu(K_w)} = \frac{C_D}{g(w)^2} \leq \frac{C_D}{\beta^2 s^2}$$

for any $w \in \Lambda_s$. Note that under the assumption of this lemma, $\lambda_1(K^I_w)$ is the smallest eigenvalue of H_w for any $w \in W_*$. Now let $\delta := \max\{C_D\beta^{-2}, 1\}$, $x \in [\delta,\infty)$ and $s^2 := \delta/x$. Since $x \geq C_D/\beta^2 s^2$, $\lambda_1(K^I_w) \leq x$ and $N_w(x) \geq 1$ for any $w \in \Lambda_s$. Hence by Lemma 4.8,

$$N_D(x) \geq \sum_{w\in\Lambda_s} N_w(x) \geq \#\Lambda_s = \#\Lambda_{\sqrt{\delta}x^{-1/2}}.$$

By Proposition 2.7, there exists $c_1 > 0$ such that $c_1\#\Lambda_{\delta^{-1/2}t} \leq \#\Lambda_t$ for any $t \in (0,1]$. Thus the result follows. ■

Proof of Theorem 4.3. H_N and H_D have compact resolvents by Lemma 4.5. Since $\mathcal{F}_{K^I} \subset \mathcal{F}$, the minimax principle shows that $N_D(x) \leq N_N(x)$ for any $x \in [0,\infty)$. Now the statement is immediate from Lemmas 4.6 and 4.9. $\blacksquare_{\textbf{Theorem 4.3}}$

5. Short time asymptotics of the partition function

In this section we assume that $(\mathcal{L} = (K,S,\{F_i\}_{i\in S}), \mu, \mathcal{E}, \mathcal{F}, \mathbf{r} = (r_i)_{i\in S})$ is a self-similar Dirichlet space and that $\mathcal{S} = \{\Lambda_s\}_{s\in(0,1]}$ is the scale induced by the gauge function $g : w \mapsto \sqrt{r_w \mu(K_w)}$. We also assume throughout this section that μ is a self-similar measure with weight $(\mu_i)_{i\in S}$. In particular, $\mathcal{S}$ is a self-similar scale with weight $\boldsymbol{\gamma} = (\gamma_i)_{i\in S}$, where $\gamma_i := \sqrt{r_i \mu_i}$. We set $d_{\mathrm{S}} := d(\boldsymbol{\gamma})$, where $d(\boldsymbol{\gamma})$ is as in Proposition 2.9 with $\boldsymbol{\alpha} = \boldsymbol{\gamma}$. We have $d_{\mathrm{S}} > 0$ since $\#S \geq 2$, and $\dim_{\mathcal{S}} K = d_{\mathrm{S}}$ by (2.2).

Notation. Let $\{T^N_t\}_{t\in(0,\infty)}$ and $\{T^D_t\}_{t\in(0,\infty)}$ be the strongly continuous contraction semigroups associated with the closed forms $(\mathcal{E},\mathcal{F})$ on $L^2(K,\mu)$ and $(\mathcal{E}^{K^I}, \mathcal{F}_{K^I})$ on $L^2(K^I, \mu|_{K^I})$, respectively. For $b \in \{N, D\}$, let Z_b denote the partition function associated with $\{T^b_t\}_{t\in(0,\infty)}$ (recall Definition 4.1). Note that if $\{T^b_t\}_{t\in(0,\infty)}$ is ultracontractive (see Definition A.1 (1)) then by [14, Theorem 2.1.4] H_b has compact resolvent and $Z_b(t) \in (0,\infty)$ for any $t \in (0,\infty)$.

In our case, the (sub-)Gaussian heat kernel upper bound is formulated as follows.

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Definition 5.1 (UHK) We say that the *(sub-)Gaussian heat kernel upper bound* holds for $(\mathcal{L},\mu,\mathcal{E},\mathcal{F},\mathbf{r})$, or simply (UHK) holds, if and only if the following conditions are valid: The semigroup $\{T_t^N\}_{t\in(0,\infty)}$ has a heat kernel $\{p_t^N\}_{t\in(0,\infty)}$, and there exist $\beta\in(1,\infty)$, a $(2/\beta)$-qdistance d adapted to $\mathcal{S}$ and $c_1,c_2\in(0,\infty)$ such that for each $t\in(0,1]$,

$$p_t^N(x,y)\leq\frac{c_1}{\mu(B_{\sqrt{t}}(x,d))}\exp\biggl(-c_2\Bigl(\frac{d(x,y)^2}{t}\Bigr)^{\frac{1}{\beta-1}}\biggr)\qquad\mu\times\mu\text{-a.e. }(x,y)\in K\times K.\quad(5.1)$$

If $\dim_{\mathcal{S}}\overline{V_0}\leq d_\partial$ for some $d_\partial\in[0,d_{\mathrm{S}})$, then (UHK) leads us to the following asymptotic behavior of Z_b, which is the main theorem of this section.

Theorem 5.2 (Short time asymptotics of the partition function) *Let $d_\partial\in[0,d_{\mathrm{S}})$ and suppose that $\dim_{\mathcal{S}}\overline{V_0}\leq d_\partial$ and* (UHK) *hold. Then we have the following statements.*
(1) Non-lattice case: If $\sum_{i\in S}\mathbb{Z}\log\gamma_i$ is a dense additive subgroup of $\mathbb{R}$, then $t^{d_{\mathrm{S}}/2}Z_N(t)$ and $t^{d_{\mathrm{S}}/2}Z_D(t)$ converge as $t\downarrow 0$ and

$$\lim_{t\downarrow 0}t^{d_{\mathrm{S}}/2}Z_N(t)=\lim_{t\downarrow 0}t^{d_{\mathrm{S}}/2}Z_D(t)\in(0,\infty).\tag{5.2}$$

(2) Lattice case: If $\sum_{i\in S}\mathbb{Z}\log\gamma_i$ is a discrete additive subgroup of $\mathbb{R}$, let $T\in(0,\infty)$ be its generator. Define $m_i:=-\log\gamma_i/T(\in\mathbb{N})$ and $p_i:=\gamma_i^{d_{\mathrm{S}}}$ for each $i\in S$ and let Q be the polynomial defined by $Q(z):=(1-\sum_{i\in S}p_iz^{m_i})/(1-z)$. Set $q:=\min\{|z|\mid z\in\mathbb{C},Q(z)=0\}$ ($q:=\infty$ if $Q=1$), $m:=\max\{$the order of zero of Q at $w\mid w\in\mathbb{C},|w|=q,Q(w)=0\}$ and $d_{\mathrm{M}}:=d_{\mathrm{S}}-T^{-1}\log q$. Then there exists a continuous T-periodic function $G:\mathbb{R}\to(0,\infty)$ such that, for any $b\in\{N,D\}$, as $t\downarrow 0$,

$$Z_b(t)-t^{-d_{\mathrm{S}}/2}G\Bigl(\frac12\log\frac1t\Bigr)=\left\{\begin{array}{cl}O\bigl(t^{-d_\partial/2}\bigr) & \text{if } e^{(d_{\mathrm{S}}-d_\partial)T}<q,\\ O\bigl(t^{-d_\partial/2}\bigl(\log(t^{-1})\bigr)^m\bigr) & \text{if } e^{(d_{\mathrm{S}}-d_\partial)T}=q,\\ O\bigl(t^{-d_{\mathrm{M}}/2}\bigl(\log(t^{-1})\bigr)^{m-1}\bigr) & \text{if } e^{(d_{\mathrm{S}}-d_\partial)T}>q.\end{array}\right.\tag{5.3}$$

Remark. In the lattice case we have $q>1$, and therefore $d_{\mathrm{M}}\in(d_\partial,d_{\mathrm{S}})$ if $e^{(d_{\mathrm{S}}-d_\partial)T}>q$. In fact, $Q(1)=\sum_{i\in S}m_ip_i\geq\sum_{i\in S}p_i=1$. If $\sum_{i\in S}p_iz^{m_i}=1$ for $z\in\mathbb{C}$ with $|z|=1$, then the triangle inequality implies that $z^{m_i}=z^{m_j}$ for any $i,j\in S$. Hence $z=1$. Also clearly $\bigl|\sum_{i\in S}p_iz^{m_i}\bigr|\leq\sum_{i\in S}p_i|z|=|z|<1$ if $z\in\mathbb{C}$ and $|z|<1$. Thus $q>1$.

As a special case of the above theorem, we have the following.

Corollary 5.3 *Let $d_\partial\in[0,d_{\mathrm{S}})$ and suppose that $\dim_{\mathcal{S}}\overline{V_0}\leq d_\partial$ and* (UHK) *hold. If $\gamma_i=\gamma$ for any $i\in S$ for some $\gamma\in(0,1)$, then there exists a continuous $\log(\gamma^{-1})$-periodic function $G:\mathbb{R}\to(0,\infty)$ such that, for any $b\in\{N,D\}$, as $t\downarrow 0$,*

$$Z_b(t)-t^{-d_{\mathrm{S}}/2}G\Bigl(\frac12\log\frac1t\Bigr)=O\bigl(t^{-d_\partial/2}\bigr).\tag{5.4}$$

Proof. Since $\sum_{i\in S}\mathbb{Z}\log\gamma_i=\mathbb{Z}\log(\gamma^{-1})$, we are in the lattice case of Theorem 5.2 and $Q=1$ in the notation there. As $q=\infty>e^{-(d_{\mathrm{S}}-d_\partial)\log\gamma}$ the corollary follows by (5.3). ■

In the non-lattice case, we have the similar asymptotic behavior of N_N and N_D.

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Corollary 5.4 *Let $d_\partial \in [0, d_S)$ and suppose that $\dim_S \overline{V_0} \le d_\partial$ and (UHK) hold. If $\sum_{i\in S} \mathbb{Z} \log \gamma_i$ is a dense additive subgroup of $\mathbb{R}$, then $x^{-d_S/2} N_N(x)$ and $x^{-d_S/2} N_D(x)$ converge as $x \to \infty$ and*

$$\lim_{x\to\infty} \frac{N_N(x)}{x^{d_S/2}} = \lim_{x\to\infty} \frac{N_D(x)}{x^{d_S/2}} \in (0, \infty). \tag{5.5}$$

Proof. This is immediate from Theorem 5.2 (1) and Karamata's Tauberian theorem (see Feller [16, p.445, Theorem 2]). ■

The rest of this section is devoted to the proof of Theorem 5.2. The proof is split into several propositions and lemmas. We first give an easy lemma on the structure of $(\mathcal{E}, \mathcal{F})$.

Lemma 5.5 (SSDF1) *and* (SSDF2) *are valid with $\mathcal{F}$ in place of $\mathcal{F} \cap C(K)$. Moreover, if $w \in W_*$ then $u^w \in \mathcal{F}_{K_w^I}$ for any $u \in \mathcal{F}_{K^I}$ and $\rho_w(\mathcal{F}_{K_w^I}) = \mathcal{F}_{K^I}$.*

Remark. If $u, v : K \to \mathbb{R}$ are Borel measurable and $u = v$ μ-a.e., then for any $w \in W_*$, it easily follows from $\mu^w = \mu$ that $u^w = v^w$ μ-a.e.

Proof. Let $w \in W_*$. Since $\mu^w = \mu$, ρ_w defines a bounded linear operator on $L^2(K, \mu)$, and also on $(\mathcal{F} \cap C(K), \mathcal{E}_1)$ by (SSDF1) and (SSDF2). Let $u \in \mathcal{F}$ and choose $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{F} \cap C(K)$ so that $u_n \to u$ in $(\mathcal{F}, \mathcal{E}_1)$. Then $u_n \circ F_w \to u \circ F_w$ in $L^2(K, \mu)$. Also in $(\mathcal{F}, \mathcal{E}_1)$, $\{u_n \circ F_w\}_{n\in\mathbb{N}}$ is a Cauchy sequence and converges to some $f \in \mathcal{F}$. Hence $u \circ F_w = f \in \mathcal{F}$ and $u_n \circ F_w \to u \circ F_w$ in $(\mathcal{F}, \mathcal{E}_1)$, which also immediately yields (3.1) for $u, v \in \mathcal{F}$.

By the equalities $\rho_w(\mathcal{C}_{K_w^I}) = \mathcal{C}_{K^I}$ (by Lemma 3.6), $\int_{K_w^I} |u|^2 d\mu = \mu_w \int_{K^I} |u \circ F_w|^2 d\mu$ for $u \in L^2(K_w^I, \mu|_{K_w^I})$ and $\mathcal{E}(u \circ F_w, u \circ F_w) = r_w \mathcal{E}(u, u)$ for $u \in \mathcal{C}_{K_w^I}$, we easily see that $\rho_w(\mathcal{F}_{K_w^I}) = \mathcal{F}_{K^I}$. Finally, let $u \in \mathcal{F}_{K^I}$ and choose $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{C}_{K^I}$ so that $u_n \to u$ in $(\mathcal{F}, \mathcal{E}_1)$. Then $\{u_n^w\}_{n\in\mathbb{N}} \subset \mathcal{C}_{K_w^I}$ by Lemma 3.6 and $\int_K |u^w - u_n^w|^2 d\mu = \mu_w \int_K |u - u_n|^2 d\mu \to 0$ as $n \to \infty$. Since $r_w \mathcal{E}(u_m^w - u_n^w, u_m^w - u_n^w) = \mathcal{E}(u_m - u_n, u_m - u_n)$ for any $m, n \in \mathbb{N}$, $\{u_n^w\}_{n\in\mathbb{N}}$ converges to some $g \in \mathcal{F}_{K_w^I}$ in $\left(\mathcal{F}_{K_w^I}, \mathcal{E}_1^{K_w^I}\right)$ and then $u^w = g \in \mathcal{F}_{K_w^I}$. ■

Lemma 5.6 *Suppose that H_N has compact resolvent and let Λ be a partition of Σ. Then*

$$N_{K_\Lambda^I}(x) = \sum_{w\in\Lambda} N_D(\gamma_w^2 x) \le N_D(x) \le N_N(x) \le \sum_{w\in\Lambda} N_N(\gamma_w^2 x) \tag{5.6}$$

for any $x \in [0, \infty)$. Moreover, there exist $c_1, c_2 \in (0, \infty)$ and $\delta \in [1, \infty)$ such that for any $x \in [\delta, \infty)$,

$$c_1 x^{d_S/2} \le N_D(x) \le N_N(x) \le c_2 x^{d_S/2}. \tag{5.7}$$

Proof. Noting Proposition 2.9, Lemma 4.8 and that $\mu(K_w \cap K_v) = 0$ for $w, v \in \Lambda$ with $w \ne v$, the same arguments as in [30, Sections 2 and 6] immediately show the lemma. ■

Now we turn to estimates of partition functions. We need the following notations.

Notation. (1) We set $A^c := K \setminus A$ for $A \subset K$.
(2) Let $U \subset K$ be non-empty open. The contraction semigroup on $L^2(U, \mu|_U)$ associated with $(\mathcal{E}^U, \mathcal{F}_U)$ is denoted by $\{T_t^U\}_{t\in(0,\infty)}$. Suppose $\{T_t^U\}_{t\in(0,\infty)}$ is ultracontractive. Then its heat kernel, which exists by [14, Theorem 2.1.4] and is unique up to $\mu\times\mu$-a.e., is denoted by $\{p_t^U\}_{t\in(0,\infty)}$. We always set $p_t^U := 0$ on $K \times K \setminus U \times U$. Also, $Z_U(t) := \mathrm{Tr}(T_t^U) = \int_{K\times K} (p_{t/2}^U)^2 d(\mu\times\mu)$ $(\in (0, \infty))$ denotes the associated partition function.

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Lemma 5.7 *Suppose that* $\{T_t^N\}_{t\in(0,\infty)}$ *is ultracontractive and let* Λ *be a partition of* Σ*. Then*

$$Z_{K_\Lambda^I}(t) = \sum_{w\in\Lambda} Z_D\Big(\frac{t}{\gamma_w^2}\Big) \le Z_D(t) \le Z_N(t) \le \sum_{w\in\Lambda} Z_N\Big(\frac{t}{\gamma_w^2}\Big) \tag{5.8}$$

for any $t \in (0,\infty)$*. Moreover, there exist* $c_1, c_2 \in (0,\infty)$ *such that for any* $t \in (0,1]$*,*

$$c_1 t^{-d_S/2} \le Z_D(t) \le Z_N(t) \le c_2 t^{-d_S/2}. \tag{5.9}$$

Proof. By [28, Proposition C.1], $\{T_t^D\}_{t\in(0,\infty)}$ and $\{T_t^{K_\Lambda^I}\}_{(t\in(0,\infty)}$ are also ultracontractive. Therefore (5.8) is an immediate consequence of (5.6) and (4.2). For $t \in (0,1]$, using Proposition 2.9, letting $\Lambda := \Lambda_{\sqrt{t}}$ in (5.8) immediately leads to (5.9). ■

In the propositions below we establish important consequences of (UHK).

Remark. In the following Proposition 5.8, Lemma 5.9, Proposition 5.10 and Theorem 5.11 and their proofs, we do **not** use the assumption that μ is a self-similar measure.

Proposition 5.8 *Suppose that* (UHK) *holds. Then*
(1) The semigroup $\{T_t^N\}_{t\in(0,\infty)}$ *is ultracontractive.*
(2) $(\mathcal{L}, \mathcal{S}, \mu)$ *satisfies* (VD)*.*
(3) $(\mathcal{L}, \mathcal{S})$ *is locally finite.*
(4) Let d *be the qdistance as in Definition 5.1. Then there exists* $c_V > 0$ *such that* $c_V\mu(B_s(x,d)) \ge \mu(U_s(x,\mathcal{S}))$ *for any* $(s,x) \in (0,1]\times K$*.*

Proof. (1) Let $t \in (0,1]$. Since $x \mapsto \mu(B_{\sqrt{t}}(x,d))$ is a $(0,1]$-valued lower semicontinuous function on a compact space K, $\eta(t) := \inf_{x\in K}\mu(B_{\sqrt{t}}(x,d)) \in (0,1]$ By (UHK), $p_t^N \le c_1\eta(t)^{-1}$ $\mu\times\mu$-a.e. on $K\times K$, hence we easily see that $\|T_t^N\|_{2\to\infty} \le c_1\eta(t)^{-1}$ for $t \in (0,1]$. Also for $t \in (1,\infty)$, $\|T_t^N\|_{2\to\infty} = \|T_1^N T_{t-1}^N\|_{2\to\infty} \le \|T_1^N\|_{2\to\infty}\|T_{t-1}^N\|_{2\to 2} \le \|T_1^N\|_{2\to\infty}$. Hence the semigroup $\{T_t^N\}_{t\in(0,\infty)}$ is ultracontractive.
(2) This is proved in exactly the same way as [28, Proofs of Lemma 3.5.5 and Theorem C.3], based on Lemma 4.7 and with a few slight modifications.
(3) Since $\mathcal{S}$ is (assumed to be) elliptic, (2) and [28, Theorem 1.3.5] imply the statement.
(4) We may choose $n \in \mathbb{N}$, $\beta_1 \in (0,1]$ and $\beta_2 \in [1,\infty)$ so that (2.5) holds. Then (2) and [28, Theorem 1.3.5] imply $(\mathrm{VD})_n$. Therefore there exists $c_V > 0$ such that

$$c_V\mu(B_s(x,d)) \ge c_V\mu\big(U^{(n)}_{\beta_2^{-1}s}(x,\mathcal{S})\big) \ge \mu(U_s^{(n)}(x,\mathcal{S})) \ge \mu(U_s(x,\mathcal{S}))$$

for any $(s,x) \in (0,1]\times K$. This completes the proof. ■

Lemma 5.9 *Suppose that* (UHK) *holds and let* $\beta \in (1,\infty)$ *and a* $(2/\beta)$*-qdistance* d *be as in Definition 5.1. Let* F *and* L *be closed subsets of* K *such that* $F \subsetneqq L \subsetneqq K$*. Then there exist* $c_1, c_2 \in (0,\infty)$ *such that, with*

$$\Phi(t,x) := \frac{c_1}{\mu(U_{\sqrt{t}}(x,\mathcal{S}))}\exp\bigg(-c_2\Big(\frac{\mathrm{dist}_d(x, L\setminus F)^2}{t}\Big)^{\frac{1}{\beta-1}}\bigg), \qquad (t,x) \in (0,1]\times K, \tag{5.10}$$

for any $t \in (0,1]$*,*

$$0 \le p_t^{F^c}(x,y) - p_t^{L^c}(x,y) \le \Phi(t,x) + \Phi(t,y) \qquad \mu\times\mu\text{-a.e. } (x,y) \in F^c\times F^c, \tag{5.11}$$

$$0 \le Z_{F^c}(t) - Z_{L^c}(t) \le \int_{F^c}\Phi(t,x)d\mu(x). \tag{5.12}$$

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Proof. By Proposition 5.8 (1) and [28, Proposition C.1], $\{T_t^{F^c}\}_{t\in(0,\infty)}$ and $\{T_t^{L^c}\}_{t\in(0,\infty)}$ are ultracontractive. Therefore the heat kernels $\{p_t^{F^c}\}_{t\in(0,\infty)}$ and $\{p_t^{L^c}\}_{t\in(0,\infty)}$ exist and Z_{F^c} and Z_{L^c} are $(0,\infty)$-valued and continuous on $(0,\infty)$. Note that $0 \le p_t^{L^c} \le p_t^{F^c} \le p_t^N$ $\mu\times\mu$-a.e. for any $t\in(0,\infty)$, which follows by [28, (C.2)] and a monotone class argument.

Let $\delta>0$ and set $\mathcal{U}_\delta := \{x\in F^c \mid \mathrm{dist}_d(x, L\setminus F)<\delta\}$. Then $\mathcal{U}_\delta$ is an open subset of F^c satisfying $L\setminus F\subset\mathcal{U}_\delta$. Note that $L\setminus F$ includes the (topological) boundary of L^c in F^c. Since $(\mathcal{E}^{F^c},\mathcal{F}_{F^c})$ is a local regular Dirichlet form by Lemma 3.4, Grigor'yan's result [20, Theorem 10.4] implies that for each $t\in(0,\infty)$, for $\mu\times\mu$-a.e. $(x,y)\in L^c\times L^c$,

$$p_t^{F^c}(x,y)-p_t^{L^c}(x,y) \le \sup_{\substack{t/2\le s\le t\\ s\in\mathbb{Q}\cup\{t/2,t\}}} \mu\text{-}\operatorname*{esssup}_{u\in\mathcal{U}_\delta} p_s^{F^c}(x,u) + \sup_{\substack{t/2\le s\le t\\ s\in\mathbb{Q}\cup\{t/2,t\}}} \mu\text{-}\operatorname*{esssup}_{v\in\mathcal{U}_\delta} p_s^{F^c}(v,y). \tag{5.13}$$

(In fact, [20, Theorem 10.4] may not be true when the right-hand side of [20, (10.12)] is essentially unbounded on some compact subset. It is, however, actually valid in the present setting, since the function $t\mapsto \mu\times\mu\text{-esssup}_{K\times K}\, p_t^{F^c}$ is $[0,\infty)$-valued and non-increasing by [20, Lemmas 3.1 and 3.2].) Moreover, (UHK), Proposition 5.8 and [28, Theorem 1.3.5] imply that there exist $c_V, c_{\mathrm{VD}}\in[1,\infty)$ such that $c_V\mu(B_s(x,d))\ge\mu(U_s(x,\mathcal{S}))$ and $c_{\mathrm{VD}}\mu(U_{s/2}(x,\mathcal{S}))\ge\mu(U_s(x,\mathcal{S}))$ for any $(s,x)\in(0,1]\times K$.

Let $t\in(0,1]$ and $s\in[t/2,t]$. By (UHK), with $c_1,c_2\in(0,\infty)$ as in Definition 5.1, for $\mu\times\mu$-a.e. $(x,u)\in L^c\times\mathcal{U}_\delta$,

$$\begin{aligned}0\le p_s^{F^c}(x,u)\le p_s^N(x,u) &\le \frac{c_1}{\mu(B_{\sqrt{s}}(x,d))}\exp\biggl(-c_2\Bigl(\frac{d(x,u)^2}{s}\Bigr)^{\frac{1}{\beta-1}}\biggr)\\ &\le \frac{c_1c_Vc_{\mathrm{VD}}}{\mu(U_{\sqrt{t}}(x,\mathcal{S}))}\exp\biggl(-c_2\Bigl(\frac{\mathrm{dist}_d(x,\mathcal{U}_\delta)^2}{t}\Bigr)^{\frac{1}{\beta-1}}\biggr)(=:\Phi(t,x,\delta)),\end{aligned}$$

which yields $\mu\text{-esssup}_{u\in\mathcal{U}_\delta}\, p_s^{F^c}(x,u)\le\Phi(t,x,\delta)$ for μ-a.e. $x\in L^c$. Thus we conclude that

$$\sup_{t/2\le s\le t,\, s\in\mathbb{Q}\cup\{t/2,t\}} \mu\text{-}\operatorname*{esssup}_{u\in\mathcal{U}_\delta} p_s^{F^c}(x,u)\le\Phi(t,x,\delta)\qquad \mu\text{-a.e. } x\in L^c.$$

Also, by the symmetry of $p_s^{F^c}$, i.e. $p_s^{F^c}(x,y)=p_s^{F^c}(y,x)$ for $\mu\times\mu$-a.e. $(x,y)\in K\times K$,

$$\sup_{t/2\le s\le t,\, s\in\mathbb{Q}\cup\{t/2,t\}} \mu\text{-}\operatorname*{esssup}_{v\in\mathcal{U}_\delta} p_s^{F^c}(v,y)\le\Phi(t,y,\delta)\qquad \mu\text{-a.e. } y\in L^c.$$

These estimates together with (5.13) imply $p_t^{F^c}(x,y)-p_t^{L^c}(x,y)\le\Phi(t,x,\delta)+\Phi(t,y,\delta)$ for $\mu\times\mu$-a.e. $(x,y)\in L^c\times L^c$. Now we define $\Phi(t,x)$ by (5.10) with c_1 replaced by $c_1c_Vc_{\mathrm{VD}}$. Then $\lim_{\delta\downarrow 0}\Phi(t,x,\delta)=\Phi(t,x)$ for any $x\in K$. Therefore setting $\delta:=n^{-1}$ with $n\in\mathbb{N}$ and letting $n\to\infty$, we see that (5.11) holds for $\mu\times\mu$-a.e. $(x,y)\in L^c\times L^c$. On the other hand, for $\mu\times\mu$-a.e. $(x,y)\in F^c\times(L\setminus F)$,

$$\begin{aligned}0\le p_t^{F^c}(x,y)-p_t^{L^c}(x,y)=p_t^{F^c}(x,y)&\le\frac{c_1}{\mu(B_{\sqrt{t}}(x,d))}\exp\biggl(-c_2\Bigl(\frac{d(x,y)^2}{t}\Bigr)^{\frac{1}{\beta-1}}\biggr)\\ \le\frac{c_1c_Vc_{\mathrm{VD}}}{\mu(U_{\sqrt{t}}(x,\mathcal{S}))}\exp\biggl(-c_2\Bigl(\frac{\mathrm{dist}_d(x,L\setminus F)^2}{t}\Bigr)^{\frac{1}{\beta-1}}\biggr)&=\Phi(t,x)\le\Phi(t,x)+\Phi(t,y).\end{aligned}$$

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Therefore by the symmetry of $p_t^{F^c}$ and $p_t^{L^c}$, (5.11) follows also for $\mu\times\mu$-a.e. $(x,y)\in F^c\times F^c\setminus L^c\times L^c$. Moreover, (5.11) and the symmetry of $p_{t/2}^{F^c}$ yield

$$\begin{aligned}
0\le Z_{F^c}(t)-Z_{L^c}(t)&=\int_{F^c\times F^c}\bigl(p_{t/2}^{F^c}(x,y)^2-p_{t/2}^{L^c}(x,y)^2\bigr)d(\mu\times\mu)(x,y)\\
&=\int_{F^c\times F^c}\bigl(p_{t/2}^{F^c}(x,y)+p_{t/2}^{L^c}(x,y)\bigr)\bigl(p_{t/2}^{F^c}(x,y)-p_{t/2}^{L^c}(x,y)\bigr)d(\mu\times\mu)(x,y)\\
&\le 2\int_{F^c\times F^c}p_{t/2}^{F^c}(x,y)\bigl(\Phi(t/2,x)+\Phi(t/2,y)\bigr)d(\mu\times\mu)(x,y)\\
&=4\int_{F^c}\int_{F^c}p_{t/2}^{F^c}(x,y)\Phi(t/2,x)d\mu(y)d\mu(x)\le 4\int_{F^c}\Phi(t/2,x)d\mu(x),
\end{aligned}$$

where we used the fact that $\int_{F^c}p_{t/2}^{F^c}(\cdot,y)d\mu(y)\le 1$ μ-a.e. on F^c. Now $\mu(U_{\sqrt{t}}(x,\mathcal{S}))\le c_{\mathrm{VD}}\mu(U_{\sqrt{t/2}}(x,\mathcal{S}))$ leads to (5.12). ■

Proposition 5.10 *Assume that $(\mathcal{L},\mathcal{S})$ is locally finite. Let $d_\partial\in[0,\infty)$, $\beta\in(1,\infty)$ and d be a $(2/\beta)$-qdistance adapted to $\mathcal{S}$. Let $A\subset K$ be non-empty and suppose $\dim_{\mathcal{S}}A\le d_\partial$. Let $c_1,c_2\in(0,\infty)$. Then there exists $c\in(0,\infty)$ such that for any $t\in(0,1]$,*

$$\int_K\frac{c_1}{\mu(U_{\sqrt{t}}(x,\mathcal{S}))}\exp\biggl(-c_2\Bigl(\frac{\mathrm{dist}_d(x,A)^2}{t}\Bigr)^{\frac{1}{\beta-1}}\biggr)d\mu(x)\le ct^{-d_\partial/2}. \tag{5.14}$$

Combining Proposition 5.10 with Proposition 5.8 (3) and (5.12), we have the following estimate, which is the key for the proof of Theorem 5.2.

Theorem 5.11 (Key estimate) *Suppose that (UHK) holds. Let F and L be closed subsets of K such that $F\subset L\subsetneqq K$. Let $d_\partial\in[0,\infty)$ and suppose $\dim_{\mathcal{S}}(L\setminus F)\le d_\partial$. Then there exists $c\in(0,\infty)$ such that for any $t\in(0,1]$,*

$$0\le Z_{F^c}(t)-Z_{L^c}(t)\le ct^{-d_\partial/2}. \tag{5.15}$$

Proof of Proposition 5.10. First let $s\in(0,1]$ and $w\in\Lambda_s$. Choose $x_0\in K_w\setminus F_w(V_0)\,(\neq\emptyset)$. Then for any $x\in K_w\setminus F_w(V_0)$, $K_s(x,\mathcal{S})=K_w=K_s(x_0,\mathcal{S})$ and $U_s(x,\mathcal{S})=\bigcup_{v\in\Lambda_{s,w}}K_v=U_s(x_0,\mathcal{S})$. Since $\mu(F_w(V_0))=0$, we have

$$\int_{K_w}\frac{1}{\mu(U_s(x,\mathcal{S}))}d\mu(x)=\int_{K_w\setminus F_w(V_0)}\frac{1}{\mu(U_s(x_0,\mathcal{S}))}d\mu(x)=\frac{\mu(K_s(x_0,\mathcal{S}))}{\mu(U_s(x_0,\mathcal{S}))}\le 1. \tag{5.16}$$

Choose $n\in\mathbb{N}$, $\beta_1\in(0,1]$ and $\beta_2\in[1,\infty)$ so that (2.5) holds. Let $s\in(0,1]$ and set $c_A:=\sup_{s\in(0,1]}s^{d_\partial}\#W(\Lambda_s,A)$ ($<\infty$ by $\dim_{\mathcal{S}}A\le d_\partial$) and $M:=\sup\{\#\Lambda_{s,w}\mid s\in(0,1],w\in\Lambda_s\}$ ($<\infty$ by the local finiteness of $(\mathcal{L},\mathcal{S})$). For $0\le k\le n$, we set $\Lambda_{s,A}^k:=W^{(k)}(\Lambda_s,A)$ (recall Definition 2.15 (2)). Then for $0\le k\le n-1$, since $\Lambda_{s,A}^{k+1}=W\bigl(\Lambda_s,\bigcup_{w\in\Lambda_{s,A}^k}K_w\bigr)=\bigcup_{w\in\Lambda_{s,A}^k}\Lambda_{s,w}$, we have $\#\Lambda_{s,A}^{k+1}\le M\#\Lambda_{s,A}^k$. Therefore

$$\#\Lambda_{s,A}^n\le M^n\#\Lambda_{s,A}^0=M^n\#W(\Lambda_s,A)\le c_AM^ns^{-d_\partial}. \tag{5.17}$$

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Let $K_s(A) := \bigcup_{w\in\Lambda^n_{s,A}} K_w \bigl(= \bigcup_{x\in A} U^{(n)}_s(x,\mathcal{S})\bigr)$. If $x \in K$ and $\mathrm{dist}_d(x,A) < \beta_1 s$, then $d(x,y) < \beta_1 s$ for some $y \in A$. Hence $x \in B_{\beta_1 s}(y,d) \subset U^{(n)}_s(y,\mathcal{S}) \subset K_s(A)$. Therefore

$$\mathrm{dist}_d(x,A) \geq \beta_1 s, \qquad x \in K \setminus K_s(A). \tag{5.18}$$

Recall that $\mathcal{S}$ is (assumed to be) elliptic. Therefore we may choose $c_3 \in (1,\infty)$ so that $g(w) \leq s \leq c_3 g(w)$ for any $s \in (0,1]$ and any $w \in \Lambda_s$. We also easily see that there exists $c_4,\gamma \in (1,\infty)$ such that $g(wv) \leq c_4\gamma^{-|v|}g(w)$ for any $w,v \in W_*$. Moreover, by Proposition 2.7 there exist $c_{\mathcal{S}},\alpha \in (0,\infty)$ such that

$$\#\Lambda_s \leq c_{\mathcal{S}} s^{-\alpha}, \qquad s \in (0,1]. \tag{5.19}$$

Let $N := N(s) := \max\{k \in \mathbb{N}\cup\{0\} \mid 2^k s \leq 1\}$, and for $0 \leq k \leq N$ let $\psi^k_s : \Lambda_s \to \Lambda_{2^k s}$ be the natural surjection, so that $w \leq \psi^k_s(w)$ for any $w \in \Lambda_s$. Let $0 \leq k \leq N$ and $w \in \Lambda_{2^k s}$. To estimate $\#\bigl((\psi^k_s)^{-1}(w)\bigr)$, let $v \in (\psi^k_s)^{-1}(w)$. Then $v \leq w$, $g(w) \leq 2^k s \leq c_3 g(w)$, $g(v) \leq s \leq c_3 g(v)$ and $g(v) \leq c_4\gamma^{-(|v|-|w|)}g(w)$. Therefore $\gamma^{|v|-|w|} \leq c_4 g(w)/g(v) \leq c_4 2^k s c_3 s^{-1} = c_3c_4 2^k$, hence $|v|-|w| \leq \lfloor (k\log 2 + \log c_3c_4)/\log\gamma \rfloor (=: \ell_k)$, where $\lfloor a \rfloor := \max\{j \in \mathbb{Z} \mid j \leq a\}$ for $a \in \mathbb{R}$. Hence by setting $c_5 := (\#S)^{1+(\log c_3c_4)/\log\gamma}/(\#S-1)$ and $\Gamma := 2^{(\log \#S)/\log\gamma}$ we have $\#\bigl((\psi^k_s)^{-1}(w)\bigr) \leq (\#S)^{\ell_k+1}/(\#S-1) \leq c_5\Gamma^k$ for any $w \in \Lambda_{2^k s}$. Then by (5.17),

$$\begin{aligned}\#\bigl((\psi^k_s)^{-1}(\Lambda^n_{2^k s,A})\bigr) &= \sum_{w\in\Lambda^n_{2^k s,A}} \#\bigl((\psi^k_s)^{-1}(w)\bigr) \leq c_5\Gamma^k \#\Lambda^n_{2^k s,A} \\ &\leq c_5\Gamma^k c_A M^n (2^k s)^{-d_\partial} = c_5 c_A M^n (2^{-d_\partial}\Gamma)^k s^{-d_\partial}.\end{aligned} \tag{5.20}$$

Note also that

$$K_{2^k s}(A) = \bigcup_{w\in\Lambda^n_{2^k s,A}} K_w = \bigcup_{w\in\Lambda^n_{2^k s,A}} \bigcup_{v\in(\psi^k_s)^{-1}(w)} K_v = \bigcup_{w\in(\psi^k_s)^{-1}(\Lambda^n_{2^k s,A})} K_w. \tag{5.21}$$

Now let $t \in (0,1]$, $N := N(\sqrt{t})$ and let $\Phi(t,x)$, $(t,x) \in (0,1]\times K$, be the integrand in the left hand side of (5.14). Since $2^{N+1}\sqrt{t} > 1$, the observations (5.16), (5.17), (5.18), (5.20), (5.21) and (5.19) yield the following estimate:

$$\begin{aligned}&\int_K \Phi(t,x)d\mu(x)\\ &= \int_{K_{\sqrt{t}}(A)} \Phi(t,x)d\mu(x) + \sum_{0<k\leq N} \int_{K_{2^k\sqrt{t}}(A)\setminus K_{2^{k-1}\sqrt{t}}(A)} \Phi(t,x)d\mu(x) + \int_{K\setminus K_{2^N\sqrt{t}}(A)} \Phi(t,x)d\mu(x)\\ &\leq \int_{K_{\sqrt{t}}(A)} \frac{c_1}{\mu(U_{\sqrt{t}}(x,\mathcal{S}))}d\mu(x) + \sum_{0<k\leq N} \int_{K_{2^k\sqrt{t}}(A)\setminus K_{2^{k-1}\sqrt{t}}(A)} \frac{c_1\exp\Bigl(-c_2\beta_1^{\frac{2}{\beta-1}}4^{\frac{k-1}{\beta-1}}\Bigr)}{\mu(U_{\sqrt{t}}(x,\mathcal{S}))}d\mu(x)\\ &\qquad + \int_{K\setminus K_{2^N\sqrt{t}}(A)} \frac{c_1}{\mu(U_{\sqrt{t}}(x,\mathcal{S}))}\exp\Bigl(-c_2(\beta_1^2/4)^{\frac{1}{\beta-1}}t^{\frac{-1}{\beta-1}}\Bigr)d\mu(x)\\ &\leq \sum_{w\in\Lambda^n_{\sqrt{t},A}} \int_{K_w} \frac{c_1}{\mu(U_{\sqrt{t}}(x,\mathcal{S}))}d\mu(x) + \sum_{\substack{0<k\leq N\\ w\in(\psi^k_{\sqrt{t}})^{-1}(\Lambda^n_{2^k\sqrt{t},A})}} \int_{K_w} \frac{c_1\exp\Bigl(-c_2\beta_1^{\frac{2}{\beta-1}}4^{\frac{k-1}{\beta-1}}\Bigr)}{\mu(U_{\sqrt{t}}(x,\mathcal{S}))}d\mu(x)\end{aligned}$$

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$$
\begin{aligned}
&\qquad + \sum_{w\in\Lambda_{\sqrt{t}}} \int_{K_w} \frac{c_1}{\mu(U_{\sqrt{t}}(x,\mathcal{S}))} \exp\Bigl(-c_2(\beta_1^2/4)^{\frac{1}{\beta-1}} t^{\frac{-1}{\beta-1}}\Bigr) d\mu(x) \\
&\le c_1 \#\Lambda^n_{\sqrt{t},A} + \sum_{0<k\le N} c_1 \exp\Bigl(-c_2\beta_1^{\frac{2}{\beta-1}} 4^{\frac{k-1}{\beta-1}}\Bigr) \#\bigl((\psi^k_{\sqrt{t}})^{-1}(\Lambda^n_{2^k\sqrt{t},A})\bigr) \\
&\qquad + c_1 \exp\Bigl(-c_2(\beta_1^2/4)^{\frac{1}{\beta-1}} t^{\frac{-1}{\beta-1}}\Bigr) \#\Lambda_{\sqrt{t}} \\
&\le c_1 c_A M^n t^{-d_\partial/2} + \sum_{0<k\le N} c_1 c_5 c_A M^n (2^{-d_\partial}\Gamma)^k \exp\Bigl(-c_2\beta_1^{\frac{2}{\beta-1}} 4^{\frac{k-1}{\beta-1}}\Bigr) t^{-d_\partial/2} \\
&\qquad + c_1 c_8 t^{-\alpha/2} \exp\Bigl(-c_2(\beta_1^2/4)^{\frac{1}{\beta-1}} t^{\frac{-1}{\beta-1}}\Bigr) \\
&\le c t^{-d_\partial/2},
\end{aligned}
$$

where $c \in (0,\infty)$ is a constant determined solely by the constants given in the assumptions. Thus the proof is complete. $\blacksquare_{\textbf{Proposition 5.10}}$

Proof of Theorem 5.2. Since $Z_N = Z_K = Z_{\emptyset^c}$ and $Z_D = Z_{K^I} = Z_{(\overline{V_0})^c}$, Theorem 5.11 implies that there exists $c_0 \in (0,\infty)$ such that for any $t \in (0,1]$,

$$0 \le Z_N(t) - Z_D(t) \le c t^{-d_\partial/2}. \tag{5.22}$$

Let $\gamma := \min_{i\in S}\gamma_i$. By (5.8), for any $t \in (0,\gamma^2]$,

$$0 \le Z_D(t) - \sum_{i\in S} Z_D\Bigl(\frac{t}{\gamma_i^2}\Bigr) \le \sum_{i\in S}\Bigl(Z_N\Bigl(\frac{t}{\gamma_i^2}\Bigr) - Z_D\Bigl(\frac{t}{\gamma_i^2}\Bigr)\Bigr) \le c_0(\#S)t^{-d_\partial/2}.$$

On the other hand, $0 \le Z_D(t) - \sum_{i\in S} Z_D\bigl(t\gamma_i^{-2}\bigr) \le Z_D(t) \le Z_D(\gamma^2)$ for any $t \in [\gamma^2,\infty)$. Therefore if we set $c_Z := \max\{c_0(\#S), Z_D(\gamma^2)\}$, then

$$0 \le Z_D(t) - \sum_{i\in S} Z_D\Bigl(\frac{t}{\gamma_i^2}\Bigr) \le c_Z t^{-d_\partial/2}, \qquad t \in (0,1]. \tag{5.23}$$

Define $\Psi_D(x) := \max\{0, Z_D(x^{-1}) - Z_D(1)\}$ for each $x \in (0,\infty)$. Then $\Psi_D(x) = 0$ for any $x \in (0,1]$. Moreover, by (5.23) we easily see that

$$0 \le \Psi_D(x) - \sum_{i\in S} \Psi_D(\gamma_i^2 x) \le c x^{d_\partial/2} \tag{5.24}$$

for any $x \in (0,\infty)$, with a different constant $c \in (0,\infty)$.

We closely follow [27, Proof of Theorem 4.1.5] in the rest of this proof. Define $f(t) := e^{-d_{\mathrm{S}}t}\Psi_D(e^{2t})$ and $u(t) := e^{-d_{\mathrm{S}}t}\bigl(\Psi_D(e^{2t}) - \sum_{i\in S}\Psi_D(\gamma_i^2 e^{2t})\bigr)$ for $t \in \mathbb{R}$. f and u are bounded and continuous. Letting $p_i := \gamma_i^{d_{\mathrm{S}}}$ for $i \in S$, so that $\sum_{i\in S} p_i = 1$, we have the following *renewal equation*

$$f(t) = \sum_{i\in S} p_i f(t - \log(\gamma_i^{-1})) + u(t), \qquad t \in \mathbb{R}. \tag{5.25}$$

We have $f(t) = u(t) = 0$ for any $t \in (-\infty,0]$, and (5.24) yields $0 \le u(t) \le c e^{-(d_{\mathrm{S}}-d_\partial)t}$ for any $t \in [0,\infty)$. Since we assume that $d_{\mathrm{S}} - d_\partial > 0$, all the conditions required for the

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renewal theorem [27, Theorems B.4.2 and B.4.3] are satisfied (see also Feller [16, Chapter XI] for the renewal theorem). Thus, for the non-lattice case, we have

$$\lim_{t\to\infty} f(t) = \Bigl(\sum_{i\in S} \gamma_i^{d_S} \log(\gamma_i^{-1})\Bigr)^{-1} \int_0^\infty u(t)dt \,(\in \mathbb{R}),$$

and this means that $t^{d_S/2}Z_D(t)$ converges as $t \downarrow 0$. $\lim_{t\downarrow 0} t^{d_S/2}Z_D(t) \in (0,\infty)$ by (5.9). (5.22) implies that $t^{d_S/2}Z_N(t)$ also converges to the same limit as $t \downarrow 0$.

For the lattice case, it is clear that the series $\sum_{j\in\mathbb{Z}} u(\cdot + jT)$ is uniformly absolutely convergent on every compact subset of $\mathbb{R}$, hence the function G on $\mathbb{R}$ defined by $G(t) := \widetilde{M}\sum_{j\in\mathbb{Z}} u(t+jT)$, $t\in\mathbb{R}$, where $\bigl(\widetilde{M}\bigr)^{-1} := \sum_{i\in S} m_i p_i$, is T-periodic and continuous. By [27, Theorem B.4.3], $\lim_{t\to\infty} |G(t) - f(t)| = 0$, and this is clearly equivalent to $\lim_{t\downarrow 0} |t^{d_S/2}Z_D(t) - G(2^{-1}\log(t^{-1}))| = 0$. Then (5.9) implies that G is $(0,\infty)$-valued. Moreover, [27, Theorem B.4.3] leads also to the following estimate of $|f(t) - G(t)|$:

$$\text{As } t\to\infty,\quad |f(t)-G(t)| = \begin{cases} O\bigl(e^{-(d_S-d_\partial)t}\bigr) & \text{if } e^{(d_S-d_\partial)T} < q, \\ O\bigl(t^m e^{-(d_S-d_\partial)t}\bigr) & \text{if } e^{(d_S-d_\partial)T} = q, \\ O\bigl(t^{m-1}q^{-t/T}\bigr) & \text{if } e^{(d_S-d_\partial)T} > q. \end{cases} \tag{5.26}$$

Now all the statements for the lattice case are obvious from (5.22) and (5.26). $\blacksquare$**Theorem 5.2**

6. Rational boundary and cell-counting dimension

This and the next sections are devoted to giving some complementary statements concerning the main result of the previous section (Theorem 5.2) and its proof. In this Section 6, we provide a practical method of calculating the cell-counting dimension of self-similar subsets with respect to a self-similar scale. We also see that the inequality $\dim_S \overline{V_0} < d_S$ is valid for all typical examples.

Let S be a non-empty finite set.

Definition 6.1 Let X be a non-empty finite subset of $W_\#(= W_* \setminus \{\emptyset\})$.
(1) We write $w = (w)_1 \dots (w)_{|w|}$ for any $w \in W_\#$. We define $\iota_X : \Sigma(X) \to \Sigma = \Sigma(S)$ and $\iota_X^W : W_*(X) \to W_* = W_*(S)$ to be the natural identifications, that is,

$$\iota_X(x_1x_2\dots) := (x_1)_1\dots(x_1)_{|x_1|}(x_2)_1\dots(x_2)_{|x_2|}\dots\dots,$$
$$\iota_X^W(x_1\dots x_m) := (x_1)_1\dots(x_1)_{|x_1|}\dots(x_m)_1\dots(x_m)_{|x_m|}.$$

(2) We set $\Sigma[X] := \iota_X(\Sigma(X))$ and $\Sigma_w[X] := \sigma_w(\Sigma[X])$.
(3) X is called *independent* if and only if ι_X is injective. Clearly, If X is independent then ι_X^W is also injective. Accordingly, when X is independent, we will often identify $x_1\dots x_m \in W_*(X)$ with $\iota_X^W(x_1\dots x_m) \in W_*$ and $\Sigma(X)$ with $\Sigma[X]$ through ι_X.

Note that, for $X \subset W_\#$ non-empty finite, $\Sigma[X]$ is compact since ι_X is continuous.

Below we collect basic facts on $\Sigma_x[X]$, where $X \subset W_\#$ is non-empty finite and $x \in W_*$.

Definition 6.2 (1) Let $\Sigma_0 \subset \Sigma$ be non-empty and $x \in W_*$. For each $\omega\in\Sigma$, we define $O_{\Sigma_0,x}(\omega) := \#\bigl(\{n \in \mathbb{N}\cup\{0\} \mid \sigma^n\omega \in \sigma_x(\Sigma_0)\}\bigr)$, where we allow ∞ as a value of $O_{\Sigma_0,x}(\omega)$.
(2) Let $X \subset W_\#$ be non-empty finite and let $x \in W_\#$. We define

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$$A_{X,x}(w) := \{(z, x_0, x_1, \ldots, x_m) \mid m \in \mathbb{N} \cup \{0\},\ z \in W_*, x_0 = x,\\ x_1, \ldots, x_m \in X,\ zx_0x_1 \ldots x_m \le w < zx_0x_1 \ldots x_{m-1}\}$$

for each $w \in W_*$, with the convention that $zx_0x_1 \ldots x_{m-1} =: z$ when $m = 0$.

Definition 6.3 A subset X of $W_\#$ is called *separated* if and only if it is non-empty, finite and independent and satisfies $O_{\Sigma[X],y}(\omega) < \infty$ for any $\omega \in \Sigma$ for some $y \in W_\#$.

By [28, Lemma 1.6.3], $\sup_{w \in W_*} \#(A_{X,y}(w)) < \infty$ in this case.

The following lemma is useful for concrete examples, and is easily proved.

Lemma 6.4 *Let $S_1 \subsetneqq S$ be non-empty, let $X \subset W_\#(S_1)$ be non-empty finite and $x \in W_\#(S \setminus S_1)$. Then $\sup_{\omega \in \Sigma} O_{\Sigma[X],x}(\omega) = 1$.*

Lemma 6.5 *Let $\mathcal{S} = \{\Lambda_s\}_{s \in (0,1]}$ be a scale on Σ with gauge function l, let $X \subset W_\#$ be separated with $y \in W_\#$ as in Definition 6.3 and set $M := \sup_{w \in W_*} \#(A_{X,y}(w)) \,(< \infty)$. For $s \in (0, 1]$, define*

$$\begin{aligned} \Lambda_s[X] &:= \{w \in \Lambda_s \mid \Sigma_w \cap \Sigma[X] \neq \emptyset\},\\ \Lambda_s(X) &:= \{x_1 \ldots x_m \in W_*(X) \mid l(x_1 \ldots x_{m-1}) > s \ge l(x_1 \ldots x_m)\} \end{aligned} \tag{6.1}$$

with the convention that $l(x_1 \ldots x_{m-1}) = 2$ when $m = 0$. Then for any $s \in (0, 1]$,

$$\#\Lambda_s[X] \le \#\Lambda_s(X) \le M\#\Lambda_s[X]. \tag{6.2}$$

Proof. Let $s \in (0, 1]$ and let $x = x_1 \ldots x_m \in \Lambda_s(X)$. Since $l(x) \le s$, there exists a unique $\varphi(x) \in \Lambda_s$ such that $x \le \varphi(x)$. Clearly $\varphi(x) \in \Lambda_s[X]$, and we have a map $\varphi : \Lambda_s(X) \to \Lambda_s[X]$. Let $w \in \Lambda_s[X]$. Choose $x_1x_2 \cdots \in \Sigma_w \cap \Sigma[X]$ and let $m_0 := \min\{m \mid m \ge 0, |w| \le |x_1 \ldots x_m|\}$. Then we see that $x_1 \ldots x_{m_0} \in \Lambda_s(X)$ and $\varphi(x_1 \ldots x_{m_0}) = w$. Hence φ is surjective and $\#\Lambda_s[X] \le \#\Lambda_s(X)$. Next let $x_1 \ldots x_m \in \varphi^{-1}(w)$. Then $x_1 \ldots x_m \le w$ and $l(w) \le s < l(x_1 \ldots x_{m-1})$. Hence $w < x_1 \ldots x_{m-1}$ and $yx_1 \ldots x_m \le yw < yx_1 \ldots x_{m-1}$, namely $(\emptyset, y, x_1, \ldots, x_m) \in A_{X,y}(yw)$. Thus we have an injection $\varphi^{-1}(w) \to A_{X,y}(yw)$ defined by $x_1 \ldots x_m \mapsto (\emptyset, y, x_1, \ldots, x_m)$. Hence $\#\varphi^{-1}(w) \le \#(A_{X,y}(yw)) \le M$ and $\#\Lambda_s(X) = \sum_{w \in \Lambda_s[X]} \#\varphi^{-1}(w) \le M\#\Lambda_s[X]$. ■

Proposition 6.6 *Let $X \subset W_\#$ be separated, $\boldsymbol{\alpha} = (\alpha_i)_{i \in S} \in (0, 1)^S$ and let $d(\boldsymbol{\alpha}, X) \,(\in [0, \infty))$ be the unique $d \in \mathbb{R}$ that satisfies $\sum_{x \in X} \alpha_x^d = 1$. For each $s \in (0, 1]$ let $\Lambda_s[X]$ be as in (6.1) with $\Lambda_s := \Lambda_s(\boldsymbol{\alpha})$ (recall Definition 2.8). Then there exist $c_1, c_2 \in (0, \infty)$ such that for any $s \in (0, 1]$,*

$$c_1 s^{-d(\boldsymbol{\alpha},X)} \le \#\Lambda_s[X] \le c_2 s^{-d(\boldsymbol{\alpha},X)}. \tag{6.3}$$

Proof. Let $l := g_{\boldsymbol{\alpha}}$ (recall Definition 2.8), and for $s \in (0, 1]$ let $\Lambda_s(X)$ be as in (6.1). Since $\{\Lambda_s(X)\}_{s \in (0,1]}$ is a self-similar scale on $\Sigma(X)$ with weight $(\alpha_x)_{x \in X}$, Proposition 2.9 implies the existence of $c_2 \in [1, \infty)$ such that $s^{-d(\boldsymbol{\alpha},X)} \le \#\Lambda_s(X) \le c_2 s^{-d(\boldsymbol{\alpha},X)}$ for any $s \in (0, 1]$. Then Lemma 6.5 implies the assertion. ■

Let $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ be a self-similar structure in the rest of this section.

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Notation. Let $\pi : \Sigma = \Sigma(S) \to K$ denote the canonical projection associated with $\mathcal{L}$. For non-empty finite $X \subset W_{\#}$ and $w \in W_*$, we set $K[X] := \pi\big(\Sigma[X]\big)$ and $K_w[X] := \pi\big(\Sigma_w[X]\big)$.

Proposition 6.7 *Let $N \in \mathbb{N}$ and let $X_k \subset W_{\#}$ be separated and $w_k \in W_*$ for each $k = 1, \ldots, N$. Set $\Gamma := \bigcup_{k=1}^{N} \Sigma_{w_k}[X_k]$ and $L := \bigcup_{k=1}^{N} K_{w_k}[X_k]\,(= \pi(\Gamma))$. Let $\boldsymbol{\alpha} = (\alpha_i)_{i\in S} \in (0,1)^S$ and set $d_k(\boldsymbol{\alpha}) := d(\boldsymbol{\alpha}, X_k)$ for $k = 1, \ldots, N$ and $d_\Gamma(\boldsymbol{\alpha}) := \max_{k\in\{1,\ldots,N\}} d_k(\boldsymbol{\alpha})$.*
(1) If either $(\mathcal{L}, \mathcal{S}(\boldsymbol{\alpha}))$ is locally finite or $\pi^{-1}(L) = \Gamma$, then $\dim_{\mathcal{S}(\boldsymbol{\alpha})} L = d_\Gamma(\boldsymbol{\alpha})$.
(2) Let $\nu_{\boldsymbol{\alpha}}$ be the Bernoulli measure on Σ with weight $\big(\alpha_i^{d(\boldsymbol{\alpha})}\big)_{i\in S}$. Then $d_\Gamma(\boldsymbol{\alpha}) < d(\boldsymbol{\alpha})$ if and only if $\nu_{\boldsymbol{\alpha}}(\Gamma) = 0$.

In most typical cases, $V_0 = \pi(\mathcal{P}_{\mathcal{L}})$ is written in the form of Γ in Proposition 6.7. Considering such situations, we set the following definition.

Definition 6.8 (Rational boundary) We say that $\mathcal{L}$ is *of rational boundary*, or simply (RB) holds, if and only if there exist $N \in \mathbb{N}$ and a separated set $X_k \subset W_{\#}$ and $w_k \in W_*$ for each $k = 1, \ldots, N$, such that $\mathcal{P}_{\mathcal{L}} = \bigcup_{k=1}^{N} \Sigma_{w_k}[X_k]$.

Roughly speaking, (RB) says that the boundary V_0 is a finite union of self-similar sets. (RB) implies that $V_0 = \bigcup_{i=1}^{N} K_{w_k}[X_k]$, which is clearly compact, hence that $\overline{V_0} = V_0$.

When (RB) holds, we can explicitly calculate $\dim_{\mathcal{S}(\boldsymbol{\alpha})} V_0$ as in the following theorem.

Theorem 6.9 (Cell-counting dimension for rational boundaries) *Assume* (RB)*. Let $\boldsymbol{\alpha} = (\alpha_i)_{i\in S} \in (0,1)^S$ and $d_\partial(\boldsymbol{\alpha}) := \max_{1\le k\le N} d(\boldsymbol{\alpha}, X_k)$ with N, X_k as in Definition 6.8. Then $\dim_{\mathcal{S}(\boldsymbol{\alpha})} V_0 = d_\partial(\boldsymbol{\alpha})$. Moreover, $d_\partial(\boldsymbol{\alpha}) < d(\boldsymbol{\alpha})$ if and only if $K \neq V_0$.*

Kigami [28, Definition 1.5.10] has introduced the notion of *rationally ramified self-similar structures* as a class of self-similar sets with sufficiently good ramification structure, in order to argue the volume doubling property and the (sub-)Gaussian estimate of heat kernels on self-similar sets in a general framework. For example, any p.c.f. self-similar structure and any generalized Sierpinski carpet ([6, 7], see also [28, Section 3.4] and [25, §2]) are rationally ramified. By [28, Proof of Proposition 1.5.13 (1)], any rationally ramified self-similar structure satisfies (RB). See [28, Sections 1.5 and 1.6 and Chapter 2] for details about rationally ramified self-similar structures.

Proof of Theorem 6.9. $\pi(\mathcal{P}_{\mathcal{L}}) = V_0$ by definition, and $\pi^{-1}(V_0) = \mathcal{P}_{\mathcal{L}}$ by [27, Proposition 1.3.5 (1)]. Therefore $\dim_{\mathcal{S}(\boldsymbol{\alpha})} V_0 = d_\partial(\boldsymbol{\alpha})$ by Proposition 6.7 (1). If $K = V_0$ then by Proposition 2.9, $d_\partial(\boldsymbol{\alpha}) = \dim_{\mathcal{S}(\boldsymbol{\alpha})} V_0 = \dim_{\mathcal{S}(\boldsymbol{\alpha})} K = d(\boldsymbol{\alpha})$. Conversely, assume $K \neq V_0\,(= \overline{V_0})$. Let $\nu_{\boldsymbol{\alpha}}$ be the Bernoulli measure on Σ with weight $\big(\alpha_i^{d(\boldsymbol{\alpha})}\big)_{i\in S}$. Then $\nu_{\boldsymbol{\alpha}}\circ\pi^{-1}$ is a self-similar measure on K with the same weight. [28, Theorem 1.2.7] implies $0 = \nu_{\boldsymbol{\alpha}}\circ\pi^{-1}(V_0) = \nu_{\boldsymbol{\alpha}}(\mathcal{P}_{\mathcal{L}})$. Now Proposition 6.7 (2) yields $d_\partial(\boldsymbol{\alpha}) < d(\boldsymbol{\alpha})$. ■**Theorem 6.9**

Proof of Proposition 6.7. We write $\mathcal{S}$, Λ_s, d, d_k, d_Γ, ν and μ instead of $\mathcal{S}(\boldsymbol{\alpha})$, $\Lambda_s(\boldsymbol{\alpha})$, $d(\boldsymbol{\alpha})$, $d_k(\boldsymbol{\alpha})$, $d_\Gamma(\boldsymbol{\alpha})$, $\nu_{\boldsymbol{\alpha}}$ and $\mu_{\boldsymbol{\alpha}}$ in this proof. Set $\alpha := \min_{i\in S} \alpha_i$, $\alpha_W := \min_{1\le k\le N} \alpha_{w_k}$ and $\Lambda_s[X_k] := \{w \in \Lambda_s \mid \Sigma_w \cap \Sigma[X_k] \neq \emptyset\}$ for $k = 1, \ldots, N$ and $s \in (0,1]$, as in Lemma 6.5. Then Proposition 6.6 implies that there exist $c_1, c_2 \in (0,\infty)$ such that for any $k \in \{1, \ldots, N\}$,

$$c_1 s^{-d_k} \le \#\Lambda_s[X_k] \le c_2 s^{-d_k}, \qquad s \in (0,1]. \tag{6.4}$$

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(1) Since $1 \leq s^{-d_\Gamma} \leq \alpha_W^{-d_\Gamma}$ and $1 \leq \#W(\Lambda_s, L) \leq \#\Lambda_s \leq \alpha^{-d}s^{-d} \leq \alpha^{-d}\alpha_W^{-d}$ for $s \in [\alpha_W, 1]$, we may assume that $s \in (0, \alpha_W]$. Let $s \in (0, \alpha_W]$. Then

$$\begin{aligned} W(\Lambda_s, L) &\supset \{w \in \Lambda_s \mid \Sigma_w \cap \Gamma \neq \emptyset\} \\ &= \bigcup_{k=1}^{N} \{w \in \Lambda_s \mid \Sigma_w \cap \Sigma_{w_k}[X_k] \neq \emptyset\} = \bigcup_{k=1}^{N} \bigl\{w_k v \mid v \in \Lambda_{s\alpha_{w_k}^{-1}}[X_k]\bigr\}. \end{aligned} \tag{6.5}$$

Choose $J \in \{1, \ldots, N\}$ so that $d_J = d_\Gamma$. By (6.4) and $d_\Gamma = d_J$,

$$\#W(\Lambda_s, L) \geq \#\bigl\{w_J v \mid v \in \Lambda_{s\alpha_{w_J}^{-1}}[X_J]\bigr\} = \#\Lambda_{s\alpha_{w_J}^{-1}}[X_J] \geq c_1 \alpha_{w_J}^{d_J} s^{-d_\Gamma}. \tag{6.6}$$

To estimate $\#W(\Lambda_s, L)$ from above, let $M := 1$ if $\pi^{-1}(L) = \Gamma$ and otherwise let $M := \sup\{\#\Lambda_{t,w} \mid t \in (0,1], w \in \Lambda_t\}$ ($< \infty$ by the assumption). Then we have $\#W(\Lambda_s, L) \leq M\#\{w \in \Lambda_s \mid \Sigma_w \cap \Gamma \neq \emptyset\}$. Indeed, if $\pi^{-1}(L) = \Gamma$ then clearly $W(\Lambda_s, L) = \{w \in \Lambda_s \mid \Sigma_w \cap \Gamma \neq \emptyset\}$ and $\#W(\Lambda_s, L) = \#(\{w \in \Lambda_s \mid \Sigma_w \cap \Gamma \neq \emptyset\})$. If $\pi^{-1}(L) \neq \Gamma$, let $v \in W(\Lambda_s, L)$. Choose $x \in K_v \cap L$, $\omega \in \Gamma \cap \pi^{-1}(x)$ and $w \in \Lambda_s$ so that $\omega \in \Sigma_w$. Then $x \in K_w \cap K_v \neq \emptyset$, hence $v \in \Lambda_{s,w}$, and $\omega \in \Sigma_w \cap \Gamma \neq \emptyset$. Therefore $W(\Lambda_s, L) \subset \bigcup\{\Lambda_{s,w} \mid w \in \Lambda_s, \Sigma_w \cap \Gamma \neq \emptyset\}$ and $\#W(\Lambda_s, L) \leq M\#(\{w \in \Lambda_s \mid \Sigma_w \cap \Gamma \neq \emptyset\})$. Now by (6.4), (6.5),

$$\frac{\#W(\Lambda_s, L)}{M} \leq \sum_{k=1}^{N} \#\bigl\{w_k v \mid v \in \Lambda_{s\alpha_{w_k}^{-1}}[X_k]\bigr\} = \sum_{k=1}^{N} \#\Lambda_{s\alpha_{w_k}^{-1}}[X_k] \leq \sum_{k=1}^{N} c_2 \alpha_{w_k}^{d_k} s^{-d_k} \leq cs^{-d_\Gamma},$$

where $c := \sum_{k=1}^{N} c_2 \alpha_{w_k}^{d_k}$, and $\dim_{\mathrm{S}} L = d_\Gamma$ follows from this and (6.6).

(2) Let $k \in \{1, \ldots, N\}$. Note that $\Sigma_s[X_k] := \bigcup_{w \in \Lambda_s[X_k]} \Sigma_w$, $s \in (0, 1]$, is decreasing as $s \downarrow 0$ and that $\bigcap_{s \in (0,1]} \Sigma_s[X_k] = \Sigma[X_k]$. Also, $\nu(\Sigma[X_k]) = \alpha_{w_k}^{-d}\nu(\Sigma_{w_k}[X_k])$. Now since

$$c_1 \alpha^d s^{d-d_k} \leq \alpha^d s^d \#\Lambda_s[X_k] \leq \sum_{w \in \Lambda_s[X_k]} \alpha_w^d = \nu(\Sigma_s[X_k]) \leq s^d \#\Lambda_s[X_k] \leq c_2 s^{d-d_k},$$

$$\text{we have} \quad \limsup_{s \downarrow 0} c_1 \alpha^d s^{d-d_k} \leq \nu(\Sigma[X_k]) = \alpha_{w_k}^{-d}\nu(\Sigma_{w_k}[X_k]) \leq \liminf_{s \downarrow 0} c_2 s^{d-d_k}.$$

Hence $d_k < d$ if and only if $\nu(\Sigma_{w_k}[X_k]) = 0$, and the statement follows. ■**Proposition 6.7**

7. Sharpness of the key estimate

In this section, we prove a better lower bound for (5.15) in Theorem 5.11 in terms of the cell-counting dimension of $L \setminus F$, under the condition that $L \setminus F$ includes a self-similar subset of positive capacity. This shows a sharpness of the upper bound in (5.15).

For this purpose, we need the notion of *intersection type* introduced by Kigami [28, Section 2.2]. Subsection 7.1 is devoted to a brief description of basic facts on intersection type. The statement and the proof of sharpness of the key estimate is provided in Subsection 7.2 (Theorem 7.7). The proof of Theorem 7.7 relies heavily on strict positivity of heat kernels and of hitting probabilities, which is separately argued in the appendix in the framework of a general regular Dirichlet form. In Subsection 7.3 we establish a reasonable sufficient condition for the positivity of capacity (Theorem 7.18), which plays an essential role in applying Theorem 7.7 to generalized Sierpinski carpets in Section 8.

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7.1. Intersection type

Throughout this subsection, we fix a self-similar structure $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ and a scale $\mathcal{S} = \{\Lambda_s\}_{s\in(0,1]}$ on $\Sigma = \Sigma(S)$. We state basic definitions on intersection type only briefly. See Kigami [28, Section 2.2] for basic facts about intersection type.

Definition 7.1 (1) Define $\mathcal{IP}(\mathcal{L}) := \{(w, v) \mid w, v \in W_\#, K_w \cap K_v \neq \emptyset, \Sigma_w \cap \Sigma_v = \emptyset\}$. Each $(w, v) \in \mathcal{IP}(\mathcal{L})$ is called an *intersection pair of* $\mathcal{L}$.
(2) Set $\mathcal{A} := \{(A, B, \varphi) \mid A, B \subset V_0$ non-empty compact, $\varphi : A \to B$ homeomorphism$\}$. For each $(w, v) \in \mathcal{IP}(\mathcal{L})$, we define $\Phi_{\mathcal{IT}}((w, v)) \in \mathcal{A}$ by

$$\Phi_{\mathcal{IT}}((w, v)) := \bigl(F_w^{-1}(K_w \cap K_v), F_v^{-1}(K_w \cap K_v), F_v^{-1}\circ F_w|_{F_w^{-1}(K_w\cap K_v)}\bigr).$$

Definition 7.2 (Intersection type) (1) We set $\mathcal{IT}(\mathcal{L}) := \Phi_{\mathcal{IT}}(\mathcal{IP}(\mathcal{L}))$. Each element of $\mathcal{IT}(\mathcal{L})$ is called an *intersection type of* $\mathcal{L}$.
(2) We define $\mathcal{IP}(\mathcal{L}, \mathcal{S}) := \{(w, v) \mid w, v \in \Lambda_s$ for some $s \in (0, 1]$ and $(w, v) \in \mathcal{IP}(\mathcal{L})\}$ and $\mathcal{IT}(\mathcal{L}, \mathcal{S}) := \Phi_{\mathcal{IT}}(\mathcal{IP}(\mathcal{L}, \mathcal{S}))$. We say that $\mathcal{S}$ is *intersection type finite with respect to* $\mathcal{L}$, or simply $(\mathcal{L}, \mathcal{S})$ is *intersection type finite*, if and only if $\#\mathcal{IT}(\mathcal{L}, \mathcal{S}) < \infty$.

Definition 7.3 (1) A non-empty finite subset Γ of W_* is said to be a *sub-partition of* Σ if and only if $\Sigma_w \cap \Sigma_v = \emptyset$ for any $w, v \in \Gamma$ with $w \neq v$.
(2) Let $\Gamma_1, \Gamma_2 \subset W_*$ be sub-partitions of Σ. A bijection $\varphi : \Gamma_1 \to \Gamma_2$ is called an $\mathcal{L}$-*isomorphism* if and only if φ possesses the following two properties:
(i) For $w, v \in \Gamma_1$, $(w, v) \in \mathcal{IP}(\mathcal{L})$ if and only if $(\varphi(w), \varphi(v)) \in \mathcal{IP}(\mathcal{L})$.
(ii) $\Phi_{\mathcal{IT}}((w, v)) = \Phi_{\mathcal{IT}}((\varphi(w), \varphi(v)))$ for any $w, v \in \Gamma_1$ with $(w, v) \in \mathcal{IP}(\mathcal{L})$.
(3) Let $\varphi : \Gamma_1 \to \Gamma_2$ be an $\mathcal{L}$-isomorphism between sub-partitions Γ_1, Γ_2 of Σ. We define $F_\varphi : K(\Gamma_1) \to K(\Gamma_2)$ (recall Definition 2.15 (1)) by $F_\varphi|_{K_w} := F_{\varphi(w)}\circ F_w^{-1}$ for any $w \in \Gamma_1$. F_φ is a well-defined homeomorphism. We call F_φ the $\mathcal{L}$-*similitude associated with* φ. Moreover, if μ is a self-similar measure on K and $K \neq \overline{V_0}$, define a bounded linear operator $\rho_\varphi : L^2(K(\Gamma_2), \mu|_{K(\Gamma_2)}) \to L^2(K(\Gamma_1), \mu|_{K(\Gamma_1)})$ by $\rho_\varphi u := u\circ F_\varphi$. Also for $u : K \to \mathbb{R}$, we define $u^\varphi : K \to \mathbb{R}$ by $u^\varphi := \left\{\begin{smallmatrix} u\circ F_\varphi^{-1} & \text{on} & K(\Gamma_2) \\ 0 & \text{on} & K\setminus K(\Gamma_2)\end{smallmatrix}\right.$.

Definition 7.4 Let $n \in \mathbb{N} \cup \{0\}$. For (s_1, x_1), $(s_2, x_2) \in (0, 1]\times K$, we write $(s_1, x_1) \underset{\mathcal{L},\mathcal{S}}{\overset{n}{\sim}} (s_2, x_2)$ if and only if there exists an $\mathcal{L}$-isomorphism $\varphi : \Lambda^n_{s_1,x_1} \to \Lambda^n_{s_2,x_2}$ such that $\varphi(\Lambda^k_{s_1,x_1}) = \Lambda^k_{s_2,x_2}$ for any $k = 0, \dots, n$. Such φ is called an $(n, \mathcal{L}, \mathcal{S})$-*isomorphism between* (s_1, x_1) *and* (s_2, x_2). Clearly, $\underset{\mathcal{L},\mathcal{S}}{\overset{n}{\sim}}$ is an equivalence relation on $(0, 1]\times K$. Moreover, we write $(s_1, x_1) \underset{\mathcal{L},\mathcal{S}}{\overset{n,\varphi}{\sim}} (s_2, x_2)$ if and only if $\varphi : \Lambda^n_{s_1,x_1} \to \Lambda^n_{s_2,x_2}$ is an $(n, \mathcal{L}, \mathcal{S})$-isomorphism between (s_1, x_1) and (s_2, x_2).

The following lemma is used in the next subsection.

Notation. For $n \in \mathbb{N} \cup \{0\}$ and $(s, x) \in (0, 1]\times K$, we set $V_s^{(n)}(x, \mathcal{S}) := \mathrm{int}_K\bigl(U_s^{(n)}(x, \mathcal{S})\bigr)$.

Lemma 7.5 *Let* $n \in \mathbb{N} \cup \{0\}$ *and* $(s_1, x_1), (s_2, x_2) \in (0, 1]\times K$. *If* $(s_1, x_1) \underset{\mathcal{L},\mathcal{S}}{\overset{n+1,\varphi}{\sim}} (s_2, x_2)$, *then* $F_\varphi\bigl(V_{s_1}^{(n)}(x_1, \mathcal{S})\bigr) = V_{s_2}^{(n)}(x_2, \mathcal{S})$, *where* F_φ *is the* $\mathcal{L}$-*similitude associated with* φ.

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Proof. For $i = 1, 2$, let $U_i := U^{(n+1)}_{s_i}(x_i, \mathcal{S})$. Then $U^{(n)}_{s_i}(x_i, \mathcal{S}) \subset \mathrm{int}_K U_i$ by Lemma 2.16 (1). Therefore $V^{(n)}_{s_i}(x_i, \mathcal{S}) = \mathrm{int}_{U_i}\bigl(U^{(n)}_{s_i}(x_i, \mathcal{S})\bigr)$. Since $F_\varphi : U_1 \to U_2$ is a homeomorphism and $F_\varphi\bigl(U^{(n)}_{s_1}(x_1, \mathcal{S})\bigr) = U^{(n)}_{s_2}(x_2, \mathcal{S})$, the assertion is now immediate. ■

7.2. Sharpness of the key estimate

Throughout this subsection, $(\mathcal{L} = (K, S, \{F_i\}_{i\in S}), \mu, \mathcal{E}, \mathcal{F}, \mathbf{r} = (r_i)_{i\in S})$ is a self-similar Dirichlet space with μ a self-similar measure with weight $(\mu_i)_{i\in S}$, $\mathcal{S} = \{\Lambda_s\}_{s\in(0,1]}$ is the self-similar scale with weight $\boldsymbol{\gamma} := (\gamma_i)_{i\in S}$, $\gamma_i := \sqrt{r_i\mu_i}$, and $d_{\mathrm{S}} := d(\boldsymbol{\gamma})\ (= \dim_{\mathcal{S}} K > 0)$. We follow the notations introduced in Section 5.

The following conditions are required to verify a sharp lower bound for (5.15).

Definition 7.6 (1) We say that $(\mathcal{L}, \mu, \mathcal{E}, \mathcal{F}, \mathbf{r})$ satisfies the *strong domain self-similarity* (SSDF3$\mathcal{S}$), or simply (SSDF3$\mathcal{S}$) holds, if and only if $\mathcal{F}$ has the following property:
(SSDF3$\mathcal{S}$) For any sub-partitions Γ_1, Γ_2 of Σ, any $\mathcal{L}$-isomorphism $\varphi : \Gamma_1 \to \Gamma_2$ and any $u \in \mathcal{F} \cap C(K)$ with $\mathrm{supp}_K[u] \subset \mathrm{int}_K K(\Gamma_1)$, if $u^\varphi \in C(K)$ and $\mathrm{supp}_K[u^\varphi] \subset \mathrm{int}_K K(\Gamma_2)$ then $u^\varphi \in \mathcal{F} \cap C(K)$, where u^φ is as in Definition 7.3 (3).
(2) We say that $(\mathcal{L}, \mu, \mathcal{E}, \mathcal{F}, \mathbf{r})$ is *local weight type finite*, or simply (LWTF) holds, if and only if $\{r_w/r_v \mid (w, v) \in \mathcal{IP}(\mathcal{L}, \mathcal{S})\}$ and $\{\mu_w/\mu_v \mid (w, v) \in \mathcal{IP}(\mathcal{L}, \mathcal{S})\}$ are finite.

Clearly, (SSDF3$\mathcal{S}$) is stronger than (SSDF3) (let $\Gamma_1 = \{\emptyset\}$ and $\Gamma_2 = \{i\}$, $i \in S$).

The following is the main theorem of this section. See Definition A.1 (3) for the condition (CHK), and Definition A.4 for the definition of $\mathrm{Cap}_{\mathcal{E}}$.

Theorem 7.7 (Sharpness of the key estimate) *Assume that K is connected and that $(\mathcal{E}, \mathcal{F})$ is conservative. Suppose that $(\mathcal{L}, \mathcal{S})$ is intersection type finite and that* (LWTF), (SSDF3$\mathcal{S}$), (CHK) *and* (UHK) *hold. Let $F \subset K$ be a closed subset of K, let $w \in W_*$ and let $X \subset W_\#$ be separated and satisfy $\mathrm{Cap}_{\mathcal{E}}(K[X]) > 0$. Set $L := F \cup K_w[X]$ and $d_\partial := d(\boldsymbol{\gamma}, X)$ (recall Proposition 6.6) and suppose $F \subsetneqq L \subsetneqq K$. Then there exist $c_1, c_2 \in (0, \infty)$ such that for any $t \in (0, 1]$,*

$$c_1 t^{-d_\partial/2} \le Z_{F^c}(t) - Z_{L^c}(t) \le c_2 t^{-d_\partial/2}. \tag{7.1}$$

Remark. (1) If K is a generalized Sierpinski carpet, then we can construct a conservative self-similar Dirichlet space satisfying (SSDF3$\mathcal{S}$) and (CHK). In this case, (UHK) implies (LWTF) and that $(\mathcal{L}, \mathcal{S})$ is intersection type finite. See Section 8 for details.
(2) We have $\dim_{\mathcal{S}}(L \setminus F) = d_\partial$ in the situation of Theorem 7.7. In fact, since $(\mathcal{L}, \mathcal{S})$ is locally finite by (UHK) and Proposition 5.8 (3), Propostion 6.7 (1) implies that for any $v \in W_*$, $\dim_{\mathcal{S}} K_v[X] = d_\partial$. As $K_w[X] \not\subset F$, we can choose $x \in W_*(X)$ so that $K_{wx}[X] \cap F = \emptyset$. Then $K_{wx}[X] \subset L \setminus F \subset K_w[X]$. Hence $\dim_{\mathcal{S}}(L \setminus F) = d_\partial$ follows.
(3) The lower bound in (7.1) is the essence of Theorem 7.7. In fact, since $\dim_{\mathcal{S}}(L \setminus F) = d_\partial$, the upper bound in (7.1) follows from (UHK) and Theorem 5.11.

As a corollary of Theorem 7.7, we have a sharp estimate for the reminder term in (5.4) under the condition (RB), as follows. Recall that (RB) implies $\overline{V_0} = V_0\ (\neq K)$.

Corollary 7.8 *Assume that K is connected and that $(\mathcal{E}, \mathcal{F})$ is conservative. Suppose that $(\mathcal{L}, \mathcal{S})$ is intersection type finite and that* (LWTF), (SSDF3$\mathcal{S}$), (CHK) *and* (UHK) *hold. Suppose also that $\gamma_i = \gamma$ for any $i \in S$ for some $\gamma \in (0, 1)$ and that $\mathcal{L}$ satisfies*

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(RB) *with* $N \in \mathbb{N}$ *and* $X_k \subset W_\#$ *for* $k \in \{1, \dots, N\}$ *as in Definition 6.8. Let* $d_\partial := \max_{1 \leq k \leq N} d(\boldsymbol{\gamma}, X_k)$ ($= \dim_{\mathcal{S}} V_0 \in [0, d_{\mathrm{S}})$ *by Theorem 6.9) and let* G *be the continuous* $\log(\gamma^{-1})$*-periodic function given in Corollary 5.3. If* $\mathrm{Cap}_{\mathcal{E}}(K[X_J]) > 0$ *for some* $J \in \{1, \dots, N\}$ *satisfying* $d(\boldsymbol{\gamma}, X_J) = d_\partial$*, then there exist* $c_1, c_2 \in (0, \infty)$ *such that for any* $t \in (0, 1]$,

$$c_1 t^{-d_\partial/2} \leq t^{-d_{\mathrm{S}}/2} G\Big(\frac{1}{2} \log \frac{1}{t}\Big) - Z_D(t) \leq c_2 t^{-d_\partial/2}. \tag{7.2}$$

The rest of this subsection is devoted to the proofs of Theorem 7.7 and Corollary 7.8. First we prepare easy consequences of the assumptions. In the proofs below, $\{p_t^N\}_{t \in (0,\infty)}$ always denotes the jointly continuous heat kernel of $\{T_t^N\}_{t \in (0,\infty)}$ when (CHK) holds.

Remark. In the following Lemmas 7.9–7.12 and their proofs, we do **not** use the assumption that μ is a self-similar measure.

Lemma 7.9 *Suppose that* (CHK) *and* (UHK) *hold and let* β, d, c_1 *and* c_2 *be as in Definition 5.1. Then* (5.1) *is valid for any* $(t, x, y) \in (0, 1] \times K \times K$.

Proof. This is immediate by the lower semicontinuity of $x \mapsto \mu(B_{\sqrt{t}}(x, d))$ on K. ■

See Definition A.2 for the definitions of Feller and strong Feller properties.

Lemma 7.10 *Suppose that* $(\mathcal{E}, \mathcal{F})$ *is conservative and that* (CHK) *holds. Set* $\mathcal{P}_t(x, A) := \int_A p_t^N(x, y) d\mu(y)$ *for* $(t, x) \in (0, \infty) \times K$ *and* $A \in \mathcal{B}(K)$*. Then* $\{\mathcal{P}_t\}_{t \in (0,\infty)}$ *is a* μ*-symmetric conservative strong Feller Markovian transition function on* $(K, \mathcal{B}(K))$ *whose associated Markovian semigroup on* $L^2(K, \mu)$ *is* $\{T_t^N\}_{t \in (0,\infty)}$*. Moreover, if* (UHK) *holds in addition then* $\{\mathcal{P}_t\}_{t \in (0,\infty)}$ *is Feller.*

Proof. Let $t \in (0, \infty)$. Then $\mathcal{P}_t(\cdot, K) = T_t^N \boldsymbol{1} = \boldsymbol{1}$ μ-a.e. since $(\mathcal{E}, \mathcal{F})$ is conservative, and $\mathcal{P}_t(\cdot, K) = \int_K p_t^N(\cdot, y) d\mu(y) \in C(K)$. Therefore $\mathcal{P}_t(x, K) = 1$ for any $x \in K$. Now since $\{p_t^N\}_{t \in (0,\infty)} \subset C(K \times K)$ and it is a heat kernel of $\{T_t^N\}_{t \in (0,\infty)}$, it is clear that $\{\mathcal{P}_t\}_{t \in (0,\infty)}$ is a μ-symmetric conservative strong Feller Markovian transition function on $(K, \mathcal{B}(K))$ whose associated Markovian semigroup on $L^2(K, \mu)$ is $\{T_t^N\}_{t \in (0,\infty)}$.

Next, suppose that (UHK) holds in addition. Let $c, \alpha \in (0, \infty)$ be as in Lemma 3.7 and let β, d, c_1 and c_2 be as in Definition 5.1. By Proposition 5.8 (4), there exists $c_V \in (0, \infty)$ such that $c_V \mu(B_{\sqrt{t}}(x, d)) \geq \mu(U_{\sqrt{t}}(x, \mathcal{S}))$, hence $c c_V \mu(B_{\sqrt{t}}(x, d)) \geq t^{\alpha/2}$, for any $(t, x) \in (0, 1] \times K$. Therefore by (UHK) and Lemma 7.9 we see that

$$0 \leq p_t^N(x, y) \leq c c_1 c_V t^{-\alpha/2} \exp\left(-c_2 \Big(\frac{d(x, y)^2}{t}\Big)^{\frac{1}{\beta - 1}}\right), \qquad (t, x, y) \in (0, 1] \times K \times K. \tag{7.3}$$

Now for $f \in C(K)$, by $\int_K p_t^N(\cdot, y) d\mu(y) = \boldsymbol{1}$ on K, (7.3) and the uniform continuity of f we easily see that $\lim_{t \downarrow 0} \|\mathcal{P}_t f - f\|_\infty = 0$, proving the Feller property of $\{\mathcal{P}_t\}_{t \in (0,\infty)}$. ■

Notation. As in the appendix, for a non-empty open subset U of K, let $U_\Delta := U \cup \{\Delta_U\}$ denote the one-point compactification of U.

Lemma 7.11 *Suppose that* $(\mathcal{E}, \mathcal{F})$ *is conservative and that* (CHK) *and* (UHK) *hold.*
(1) Let $\{\mathcal{P}_t\}_{t \in (0,\infty)}$ *be as in Lemma 7.10. Then there exists a conservative diffusion* $X = \big(\Omega, \mathcal{M}, \{X_t\}_{t \in [0,\infty]}, \{\mathbf{P}_x\}_{x \in K_\Delta}\big)$ *on* K *whose transition function is* $\{\mathcal{P}_t\}_{t \in (0,\infty)}$.

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(2) For $A \in \mathcal{B}(K_\Delta)$ and $\omega \in \Omega$, define $\sigma_A(\omega) := \inf\{t \in [0,\infty) \mid X_t(\omega) \in A\}$ ($\inf \emptyset := \infty$) and $\tau_A(\omega) := \sigma_{K_\Delta \setminus A}(\omega)$. Let U be a non-empty open subset of K and define $X^U_t(\omega) := \left\{ \begin{smallmatrix} X_t(\omega) & \text{if } t<\tau_U(\omega) \\ \Delta_U & \text{if } t\geq\tau_U(\omega) \end{smallmatrix} \right.$ for $t \in [0,\infty]$ and $\omega \in \Omega$. Also set $\mathbf{P}_{\Delta_U} := \mathbf{P}_{\Delta_K}$. Then the process $X^U := \bigl(\Omega, \mathcal{M}, \{X^U_t\}_{t\in[0,\infty]}, \{\mathbf{P}_x\}_{x\in U_\Delta}\bigr)$ is a diffusion on U with $\mu|_U$-symmetric strong Feller transition function $\{\mathcal{P}^U_t\}_{t\in(0,\infty)}$ whose associated Markovian semigroup on $L^2(U,\mu|_U)$ is $\{T^U_t\}_{t\in(0,\infty)}$. Moreover, $(\mathcal{E}^U, \mathcal{F}_U)$ satisfies (CHK) *with jointly continuous heat kernel $\{p^U_t\}_{t\in(0,\infty)} \subset C_b(U\times U)$, $\mathcal{P}^U_t(x,A) = \int_A p^U_t(x,y)d\mu(y)$ for any $(t,x) \in (0,\infty)\times U$ and any $A \in \mathcal{B}(U)$, and $Z_U(t) = \int_U p^U_t(x,x)d\mu(x)$ for any $t \in (0,\infty)$.*

Proof. (1) By [9, Chapter I, Theorem 9.4] and the Feller property of $\{\mathcal{P}_t\}_{t\in(0,\infty)}$, there exists a Hunt process $X = \bigl(\Omega, \mathcal{M}, \{X_t\}_{t\in[0,\infty]}, \{\mathbf{P}_x\}_{x\in K_\Delta}\bigr)$ on K with transition function $\{\mathcal{P}_t\}_{t\in(0,\infty)}$. Since $(\mathcal{E},\mathcal{F})$ is local by Lemma 3.4, [17, Theorem 4.5.4 (ii)] implies that X is a diffusion, and it is conservative since $\mathbf{P}_x[X_t \in K] = \mathcal{P}_t(x,K) = 1$ for $(t,x) \in (0,\infty)\times K$.
(2) By [17, Theorems 4.4.2 and 4.4.3], X^U is a Hunt process on U with $\mu|_U$-symmetric transition function $\{\mathcal{P}^U_t\}_{t\in(0,\infty)}$ whose associated Markovian semigroup on $L^2(U,\mu|_U)$ is $\{T^U_t\}_{t\in(0,\infty)}$. The definition of X^U immediately implies that X^U is a diffusion. Since the transition function $\{\mathcal{P}_t\}_{t\in(0,\infty)}$ of X is both Feller and strong Feller, [13, p.69, Section 1, Proof of Theorem] implies that $\{\mathcal{P}^U_t\}_{t\in(0,\infty)}$ is strong Feller. By Proposition 5.8 (1) and [28, Proposition C.1], $\{T^U_t\}_{t\in(0,\infty)}$ is ultracontractive. Therefore Proposition A.3 (1) implies that $(\mathcal{E}^U,\mathcal{F}_U)$ satisfies (CHK) with jointly continuous heat kernel $\{p^U_t\}_{t\in(0,\infty)} \subset C_b(U\times U)$ and that $\mathcal{P}^U_t(x,A) = \int_A p^U_t(x,y)d\mu(y)$ for any $(t,x) \in (0,\infty)\times U$, $A \in \mathcal{B}(U)$. Then we have $Z_U(t) = \int_{U\times U} p^U_{t/2}(x,y)^2 d\mu(y)d\mu(x) = \int_U p^U_t(x,x)d\mu(x)$ for $t \in (0,\infty)$. ■

Convention. In the situation of Lemma 7.11 (2), we set $p^U_t(x,y) := 0$ for $t \in (0,\infty)$ and $(x,y) \in K\times K \setminus U\times U$, as stated in **Notation** before Lemma 5.7. Note that, with this convention, p^U_t *may not be continuous on* $K\times K$, although it is continuous on $U\times U$. We also set $p^\emptyset_t(x,y) := 0$ for any $(t,x,y) \in (0,\infty)\times K\times K$.

Lemma 7.12 *Suppose that $(\mathcal{E},\mathcal{F})$ is conservative and that* (CHK) *and* (UHK) *hold. Let U, V be non-empty open subsets of K. Then for any $(t,x,y) \in (0,\infty)\times K\times K$,*

$$p^N_t(x,y) - p^U_t(x,y) \geq p^V_t(x,y) - p^{U\cap V}_t(x,y). \tag{7.4}$$

Proof. Let $t \in (0,\infty)$. By Lemma 7.11, p^U_t, p^V_t and $p^{U\cap V}_t$ are continuous on $U\times U$, $V\times V$ and $(U\cap V)\times(U\cap V)$, respectively. Since $p^N_t \geq p^U_t$ and $p^N_t \geq p^V_t$ on $K\times K$, (7.4) is trivial if either $x \notin U\cap V$ or $y \notin U\cap V$. Let $x,y \in U\cap V$. Then

$$\begin{aligned}
&p^N_t(x,y) - p^U_t(x,y) - p^V_t(x,y) + p^{U\cap V}_t(x,y)\\
&= \lim_{s\downarrow 0} \frac{\int_{U_s(y,\mathcal{S})} \bigl(p^N_t(x,z) - p^U_t(x,z) - p^V_t(x,z) + p^{U\cap V}_t(x,z)\bigr)d\mu(z)}{\mu(U_s(y,\mathcal{S}))}\\
&= \lim_{s\downarrow 0} \frac{\mathbf{P}_x[X_t \in U_s(y,\mathcal{S}), \tau_U \leq t] - \mathbf{P}_x[X_t \in U_s(y,\mathcal{S}), \tau_{U\cap V} \leq t < \tau_V]}{\mu(U_s(y,\mathcal{S}))}\\
&= \lim_{s\downarrow 0} \frac{\mathbf{P}_x[X_t \in U_s(y,\mathcal{S}), \tau_U \leq t] - \mathbf{P}_x[X_t \in U_s(y,\mathcal{S}), \tau_U \leq t < \tau_V]}{\mu(U_s(y,\mathcal{S}))}\\
&= \lim_{s\downarrow 0} \frac{\mathbf{P}_x[X_t \in U_s(y,\mathcal{S}), \tau_U \leq t, \tau_V \leq t]}{\mu(U_s(y,\mathcal{S}))} \geq 0.
\end{aligned}$$

Thus the result follows. ■

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The following Lemma is the key for the proof of the lower bound of (7.1).

Lemma 7.13 *Under the assumption of Theorem 7.7, let $y_0 \in W_*(X)$ and set $\Lambda_s^{wy_0}[X] := \{v \in \Lambda_s \mid \Sigma_v \cap \Sigma_{wy_0}[X] \neq \emptyset\}$ for $s \in (0,1]$. Then (recall Definitions 2.15 and 2.17)*

$$\inf\left\{\int_{K(\Lambda_{\sqrt{t},v})} \big(p_t^N(x,x) - p_t^{K\setminus K_w[X]}(x,x)\big)d\mu(x) \,\middle|\, t \in (0,\gamma_{wy_0}^2], v \in \Lambda_{\sqrt{t}}^{wy_0}[X]\right\} > 0. \quad (7.5)$$

The proof of Lemma 7.13 is given later. We first complete the proof of Theorem 7.7 using Lemma 7.13.

Proof of Theorem 7.7. We follow the notations in Lemmas 7.10 and 7.11 above. Let $\beta \in (1,\infty)$ and a $(2/\beta)$-qdistance d be as in Definition 5.1. Since $F \subsetneqq L = F \cup K_w[X]$, $K_w[X] \not\subset F$ and we may choose $y \in W_*(X)$ so that $K_{wy} \cap F = \emptyset$. If $F \neq \emptyset$, let $c_1, c_2 \in (0,\infty)$ and $\Phi(t,x)$ be as in Lemma 5.9 with F and L replaced with $\emptyset$ and F, respectively, let $c, \alpha \in (0,\infty)$ be as in Lemma 3.7 and let $\delta := 2^{-1}\inf_{x\in K_{wy}} \mathrm{dist}_d(x,F)\,(\in (0,\infty))$. Similarly to Lemma 7.9, by (5.11) and Lemma 7.11 (2), $p_t^N(x,x) - p_t^{F^c}(x,x) \leq 2\Phi(t,x)$ for any $(t,x) \in (0,1]\times K$. Therefore, with $c_3 := 2cc_1$ and $c_4 := c_2\delta^{\frac{2}{\beta-1}}$, for any $t \in (0,1]$ and any $x \in K$ satisfying $\mathrm{dist}_d(x,K_{wy}) \leq (2^{2/\beta}-1)^{\beta/2}\delta$,

$$p_t^N(x,x) - p_t^{F^c}(x,x) \leq 2\Phi(t,x) \leq c_3 t^{-\alpha/2}\exp\big(-c_4 t^{-\frac{1}{\beta-1}}\big) \quad (7.6)$$

since $\mathrm{dist}_d(x,F) \geq \delta$. If $F = \emptyset$ then $p_t^N(x,x) = p_t^{F^c}(x,x)$ for any $(t,x) \in (0,\infty)\times K$ and (7.6) is trivially valid with some $\alpha, \delta, c_3, c_4 \in (0,\infty)$. We set $\delta_\beta := (2^{2/\beta}-1)^{\beta/2}\delta$.

For each $s \in (0,1]$, set $\Lambda_s[X] := \{v \in \Lambda_s \mid \Sigma_v \cap \Sigma[X] \neq \emptyset\}$, and, as in Lemma 7.13, $\Lambda_s^{wy}[X] := \{v \in \Lambda_s \mid \Sigma_v \cap \Sigma_{wy}[X] \neq \emptyset\}$. By Proposition 6.6, there exists $c_X \in (0,\infty)$ such that $\#\Lambda_s[X] \geq c_X s^{-d_\partial}$ for any $s \in (0,1]$. Let $s \in (0,\gamma_{wy}]$. Then we easily see that $\Lambda_s^{wy}[X] = \{wyv \mid v \in \Lambda_{s\gamma_{wy}^{-1}}[X]\}$. Therefore $\#\Lambda_s^{wy}[X] = \#\Lambda_{s\gamma_{wy}^{-1}}[X] \geq c_X\gamma_{wy}^{d_\partial}s^{-d_\partial}$.

Choose $t_* \in (0,\gamma_{wy}^2]$ so that $\mathrm{diam}_d K_v \leq 2^{-\beta/2}\delta_\beta$ for any $v \in \Lambda_{\sqrt{t_*}}$, and let $t \in (0,t_*]$. We easily see that $\mathrm{dist}_d(x,K_{wy}) \leq \delta_\beta$ for any $x \in \bigcup_{v\in\Lambda_{\sqrt{t}}^{wy}[X]} K\big(\Lambda_{\sqrt{t},v}\big)$. Since $p_t^{L^c} \leq p_t^{F^c} \leq p_t^N$ and $p_t^{L^c} \leq p_t^{K\setminus K_w[X]} \leq p_t^N$ on $K\times K$, using (7.6) and Lemma 7.13 we see that

$$\begin{aligned}
&Z_{F^c}(t) - Z_{L^c}(t) = \int_K (p_t^{F^c}(x,x) - p_t^{L^c}(x,x))d\mu(x)\\
&\geq \int_{\{\mathrm{dist}_d(\cdot,K_{wy})\leq\delta_\beta\}} (p_t^{F^c}(x,x) - p_t^{L^c}(x,x))d\mu(x)\\
&\geq \int_{\{\mathrm{dist}_d(\cdot,K_{wy})\leq\delta_\beta\}} (p_t^{N}(x,x) - p_t^{L^c}(x,x))d\mu(x) - c_3t^{-\alpha/2}\exp\big(-c_4t^{-\frac{1}{\beta-1}}\big)\\
&\geq \int_{\bigcup\{K_\tau \mid \tau\in\Lambda_{\sqrt{t},v},\,^\exists v\in\Lambda_{\sqrt{t}}^{wy}[X]\}} \big(p_t^N(x,x) - p_t^{K\setminus K_w[X]}(x,x)\big)d\mu(x) - c_3t^{-\alpha/2}\exp\big(-c_4t^{-\frac{1}{\beta-1}}\big)\\
&\geq \frac{1}{M}\sum_{v\in\Lambda_{\sqrt{t}}^{wy}[X]} \int_{K(\Lambda_{\sqrt{t},v})} \big(p_t^N(x,x) - p_t^{K\setminus K_w[X]}(x,x)\big)d\mu(x) - c_3t^{-\alpha/2}\exp\big(-c_4t^{-\frac{1}{\beta-1}}\big)\\
&\geq \frac{C_X^{wy}}{M}\#\Lambda_{\sqrt{t}}^{wy}[X] - c_3t^{-\alpha/2}\exp\big(-c_4t^{-\frac{1}{\beta-1}}\big) \geq 2c_5t^{-d_\partial/2} - c_3t^{-\alpha/2}\exp\big(-c_4t^{-\frac{1}{\beta-1}}\big),
\end{aligned}$$

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where $M := \sup\{\#\Lambda_{s,v} \mid s \in (0,1], v \in \Lambda_s\}(< \infty$ since $(\mathcal{L}, \mathcal{S})$ is locally finite by Proposition 5.8 (3)), $C_X^{wy} \in (0,\infty)$ is the infimum in (7.5) and $c_5 := C_X^{wy} c_X \gamma_{wy}^{d_\partial}/(2M)$. Choose $t_0 \in (0, t_*]$ so that $c_3 t^{-\alpha/2} \exp\bigl(-c_4 t^{-\frac{1}{\beta-1}}\bigr) \leq c_5 t^{-d_\partial/2}$ for any $t \in (0, t_0]$. Then we have $Z_{F^c}(t) - Z_{L^c}(t) \geq c_5 t^{-d_\partial/2}$ for any $t \in (0, t_0]$.

To consider the case $t \in [t_0, 1]$, let $\{\lambda_n^{F^c}\}_{n\in\mathbb{N}}$ (resp. $\{\lambda_n^{L^c}\}_{n\in\mathbb{N}}$) be the eigenvalues of the non-negative self-adjoint operator associated with $(\mathcal{E}^{F^c}, \mathcal{F}_{F^c})$ (resp. $(\mathcal{E}^{L^c}, \mathcal{F}_{L^c})$), similarly to Definition 4.1 (2). Then $\lambda_n^{F^c} \leq \lambda_n^{L^c}$ for any $n \in \mathbb{N}$ by the minimax principle, and $\lambda_n^{F^c} < \lambda_n^{L^c}$ for some $n \in \mathbb{N}$ since $\sum_{n\in\mathbb{N}} e^{-\lambda_n^{F^c} t} = Z_{F^c}(t) > Z_{L^c}(t) = \sum_{n\in\mathbb{N}} e^{-\lambda_n^{L^c} t}$ for $t \in (0, t_0]$. Therefore $Z_{F^c}(t) > Z_{L^c}(t)$ for any $t \in (0,\infty)$, and $Z_{F^c} - Z_{L^c}$ is a $(0,\infty)$-valued continuous function on $(0,\infty)$. Hence we can choose $c_6 \in (0,\infty)$ so that $Z_{F^c}(t) - Z_{L^c}(t) \geq c_6 t^{-d_\partial/2}$ for any $t \in [t_0, 1]$. ■**Theorem 7.7**

Therefore it suffices for us to prove Lemma 7.13. We need to prepare a few easy lemmas. The following lemma is stated in [24, p.600] and is easily proved by using Lemma 5.5.

Lemma 7.14 $\mathrm{Cap}_{\mathcal{E}}\bigl(F_w(A)\bigr) \geq \min\{r_w^{-1}, \mu_w\}\mathrm{Cap}_{\mathcal{E}}(A)$ *for any* $w \in W_*$ *and any* $A \subset K$.

Notation. For $n \in \mathbb{N}\cup\{0\}$ and $(s,x) \in (0,1]\times K$, we set $\mathcal{C}_{s,x}^{(n)} := \mathcal{C}_{V_s^{(n)}(x,\mathcal{S})}$ and $\mathcal{F}_{s,x}^{(n)} := \mathcal{F}_{V_s^{(n)}(x,\mathcal{S})}$. We also abbreviate $\underset{\mathcal{L},\mathcal{S}}{\overset{n}{\sim}}$ to $\overset{n}{\sim}$ and $\underset{\mathcal{L},\mathcal{S}}{\overset{n,\varphi}{\sim}}$ to $\overset{n,\varphi}{\sim}$ in the rest of this subsection.

Lemma 7.15 *Suppose that* (SSDF3$\mathcal{S}$) *holds. Let* $n \in \mathbb{N}\cup\{0\}$ *and* $(s_1,x_1),(s_2,x_2) \in (0,1]\times K$ *satisfy* $(s_1,x_1) \overset{n+1,\varphi}{\sim} (s_2,x_2)$. *Then* $\rho_\varphi\bigl(\mathcal{C}_{s_2,x_2}^{(n)}\bigr) = \mathcal{C}_{s_1,x_1}^{(n)}$ *and* $\rho_\varphi\bigl(\mathcal{F}_{s_2,x_2}^{(n)}\bigr) = \mathcal{F}_{s_1,x_1}^{(n)}$, *where we regard* $\mathcal{C}_{s_i,x_i}^{(n)}$ *and* $\mathcal{F}_{s_i,x_i}^{(n)}$ *as subspaces of* $L^2\Bigl(V_{s_i}^{(n)}(x_i,\mathcal{S}), \mu|_{V_{s_i}^{(n)}(x_i,\mathcal{S})}\Bigr)$ *for* $i = 1,2$.

Proof. Recall that the $\mathcal{L}$-similitude F_φ associated with φ induces a homeomorphism $F_\varphi : V_{s_1}^{(n)}(x_1,\mathcal{S}) \to V_{s_2}^{(n)}(x_2,\mathcal{S})$ by Lemma 7.5. Let $u \in \mathcal{C}_{s_1,x_1}^{(n)}$. Then we easily see that $u^\varphi \in C(K)$ and $\mathrm{supp}_K[u^\varphi] \subset V_{s_2}^{(n)}(x_2,\mathcal{S})$. (SSDF3$\mathcal{S}$) implies $u^\varphi \in \mathcal{F}\cap C(K)$, hence $u^\varphi \in \mathcal{C}_{s_2,x_2}^{(n)}$. Therefore $u = \rho_\varphi u^\varphi \in \rho_\varphi\bigl(\mathcal{C}_{s_2,x_2}^{(n)}\bigr)$, and it follows that $\mathcal{C}_{s_1,x_1}^{(n)} \subset \rho_\varphi\bigl(\mathcal{C}_{s_2,x_2}^{(n)}\bigr)$. By $\mathcal{C}_{s_2,x_2}^{(n)} \subset \rho_{\varphi^{-1}}\bigl(\mathcal{C}_{s_1,x_1}^{(n)}\bigr)$ and $\rho_{\varphi^{-1}} = \rho_\varphi^{-1}$, we conclude that $\rho_\varphi\bigl(\mathcal{C}_{s_2,x_2}^{(n)}\bigr) = \mathcal{C}_{s_1,x_1}^{(n)}$. Since $K_w \cap V_{s_i}^{(n)}(x_i,\mathcal{S}) = \emptyset$ for $w \in \Lambda_{s_i} \setminus \Lambda_{s_i,x_i}^n$ by Lemma 2.12, $i = 1,2$, it easily follows from (SSDF2) and the self-similarity of μ that there exist $c_1, c_2 \in (0,\infty)$ such that

$$c_1\mathcal{E}_1(\rho_\varphi u, \rho_\varphi u) \leq \mathcal{E}_1(u,u) \leq c_2\mathcal{E}_1(\rho_\varphi u, \rho_\varphi u), \qquad u \in \mathcal{C}_{s_2,x_2}^{(n)}. \tag{7.7}$$

Now $\rho_\varphi\bigl(\mathcal{F}_{s_2,x_2}^{(n)}\bigr) = \mathcal{F}_{s_1,x_1}^{(n)}$ follows from (7.7) by taking the closures in the Hilbert space $(\mathcal{F}, \mathcal{E}_1)$ for the equality $\rho_\varphi\bigl(\mathcal{C}_{s_2,x_2}^{(n)}\bigr) = \mathcal{C}_{s_1,x_1}^{(n)}$. ■

Definition 7.16 Let $n \in \mathbb{N}\cup\{0\}$. For $(s_1,x_1),(s_2,x_2) \in (0,1]\times K$, we write $(s_1,x_1) \underset{*}{\overset{n,\varphi}{\sim}} (s_2,x_2)$ if and only if $\varphi : \Lambda_{s_1,x_1}^n \to \Lambda_{s_2,x_2}^n$ is an $(n,\mathcal{L},\mathcal{S})$-isomorphism between (s_1,x_1) and (s_2,x_2) such that $r_w/r_v = r_{\varphi(w)}/r_{\varphi(v)}$ and $\mu_w/\mu_v = \mu_{\varphi(w)}/\mu_{\varphi(v)}$ for any $w,v \in \Lambda_{s_1,x_1}^n$. We also write $(s_1,x_1) \underset{*}{\overset{n}{\sim}} (s_2,x_2)$ if and only if $(s_1,x_1) \underset{*}{\overset{n,\varphi}{\sim}} (s_2,x_2)$ for some $(n,\mathcal{L},\mathcal{S})$-isomorphism $\varphi : \Lambda_{s_1,x_1}^n \to \Lambda_{s_2,x_2}^n$ between (s_1,x_1) and (s_2,x_2). Clearly, $\underset{*}{\overset{n}{\sim}}$ is an equivalence relation on $(0,1]\times K$.

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Lemma 7.17 *Suppose that* $(\mathcal{L}, \mathcal{S})$ *is both locally finite and intersection type finite and that* (LWTF) *holds. Then for any* $n \in \mathbb{N} \cup \{0\}$, $\bigl((0,1] \times K\bigr)/\underset{*}{\overset{n}{\sim}}$ *is a finite set.*

Proof. Let $n \in \mathbb{N} \cup \{0\}$. As $(\mathcal{L}, \mathcal{S})$ is locally finite, $\mathcal{L}$ is strongly finite (recall Definition 2.10 (3)) by [28, Lemma 1.3.6]. Since $(\mathcal{L}, \mathcal{S})$ is also intersection type finite, [28, Theorem 2.2.13] implies that the quotient $\bigl((0,1] \times K\bigr)/\overset{n}{\sim}$ is a finite set. Therefore there exist $J \in \mathbb{N}$ and $(s_i, x_i) \in (0,1] \times K$, $i = 1, \ldots, J$, such that for any $(s,x) \in (0,1]$ we can choose $i \in \{1, \ldots, J\}$ so that $(s,x) \overset{n}{\sim} (s_i, x_i)$.

Let $M_{\mathbf{r}} := \#\{r_w/r_v \mid (w,v) \in \mathcal{IP}(\mathcal{L}, \mathcal{S})\}$ and $M_\mu := \#\{\mu_w/\mu_v \mid (w,v) \in \mathcal{IP}(\mathcal{L}, \mathcal{S})\}$. $M_{\mathbf{r}}, M_\mu \in \mathbb{N}$ by (LWTF). Let $i \in \{1, \ldots, J\}$, $w_i \in \Lambda^0_{s_i,x_i}$ and let $(s,x) \in (0,1] \times K$ satisfy $(s_i, x_i) \overset{n,\varphi}{\sim} (s,x)$. Then for $w \in \Lambda^n_{s_i,x_i}$, there are at most $M_{\mathbf{r}}^{n+1}$ (resp. M_μ^{n+1}) possibilities for the value $r_{\varphi(w)}/r_{\varphi(w_i)}$ (resp. $\mu_{\varphi(w)}/\mu_{\varphi(w_i)}$). Therefore the cardinality of the set

$$\Bigl\{\bigl(r_{\varphi(w)}/r_{\varphi(w_i)}, \mu_{\varphi(w)}/\mu_{\varphi(w_i)}\bigr)_{w \in \Lambda^n_{s_i,x_i}} \,\Big|\, \varphi \text{ is an } (n, \mathcal{L}, \mathcal{S})\text{-isomorphism} \\ \text{between } (s_i, x_i) \text{ and some } (s,x) \in (0,1] \times K\Bigr\}$$

is bounded from above by $M_i := (M_{\mathbf{r}} M_\mu)^{(n+1)\#\Lambda^n_{s_i,x_i}}$. Hence each equivalence class of $\overset{n}{\sim}$ contains at most M_i equivalence classes of $\underset{*}{\overset{n}{\sim}}$, which implies that $\#\Bigl(\bigl((0,1] \times K\bigr)/\underset{*}{\overset{n}{\sim}}\Bigr) \leq \sum_{i=1}^J M_i < \infty$. ■

Proof of Lemma 7.13. We follow the notations in Lemmas 7.10, 7.11 and 7.15 and Definition 7.16 above. We fix $n \in \mathbb{N} \setminus \{1\}$ throughout this proof. By Proposition 5.8 (3) and Lemma 7.17, there exist $J \in \mathbb{N}$ and $(s_i, x_i) \in (0,1] \times K$, $i = 1, \ldots, J$, such that for any $(s,x) \in (0,1] \times K$ we can choose $i \in \{1, \ldots, J\}$ so that $(s,x) \underset{*}{\overset{n+1}{\sim}} (s_i, x_i)$. For $i \in \{1, \ldots, J\}$, fix $w_i \in \Lambda_{s_i,x_i}$ and set $U_i := U_{s_i}(x_i, \mathcal{S})$ and $V_i := V^{(n)}_{s_i}(x_i, \mathcal{S})$. As $n \geq 2$, $K_{w_i}[X] \subset K_{w_i} \subset K\bigl(\Lambda_{s_i,w_i}\bigr) \subset U_i \subset V_i$.

Let $t \in (0, \gamma^2_{wy_0}]$ and $v \in \Lambda^{wy_0}_{\sqrt{t}}[X]$. Then $\gamma_v \leq \sqrt{t} \leq \gamma^{-1}\gamma_v$, where $\gamma := \min_{i \in S} \gamma_i$. Since $\Sigma_v \cap \Sigma_{wy_0}[X] \neq \emptyset$, we may choose $y_1 y_2 \cdots \in \Sigma(X)$ so that $wy_0 y_1 y_2 \cdots \in \Sigma_v$. Set $j := \min\{k \mid k \in \mathbb{N} \cup \{0\}, wy_0 \ldots y_k \leq v\}$, $y_v := y_0 \ldots y_j$ and $s_v := \gamma_{wy_v}$. Fix $x_v \in K_{wy_v} \setminus F_{wy_v}(V_0)$ and set $U_v := U_{s_v}(x_v, \mathcal{S})$ and $V_v := V^{(n)}_{s_v}(x_v, \mathcal{S})$. We easily see that $wy_v \in \Lambda_{s_v}$ and $1 \leq s_v^{-2} t \leq \gamma^{-2(M_X+1)}$, where $M_X := \max_{z \in X} |z|$. As $n \geq 2$, we have $K_{wy_v}[X] \subset K_{wy_v} \subset K\bigl(\Lambda_{s_v,wy_v}\bigr) = U_v \subset V_v$.

Choose $i \in \{1, \ldots, J\}$ and $\varphi : \Lambda^{n+1}_{s_v,x_v} \to \Lambda^{n+1}_{s_i,x_i}$ so that $(s_v, x_v) \underset{*}{\overset{n+1,\varphi}{\sim}} (s_i, x_i)$. Then since $\{\varphi(wy_v)\} = \varphi(\Lambda_{s_v,x_v}) = \Lambda_{s_i,x_i} \ni w_i$, we see that $\Lambda_{s_i,x_i} = \{w_i\}$ and $\varphi(wy_v) = w_i$. By (SSDF3$\mathcal{S}$), Lemmas 7.5 and 7.15 and $(s_v, x_v) \underset{*}{\overset{n+1,\varphi}{\sim}} (s_i, x_i)$, $F_\varphi : V_v \to V_i$ is a homeomorphism,

$$\begin{aligned} &\rho_\varphi\bigl(L^2(V_i, \mu|_{V_i})\bigr) = L^2(V_v, \mu|_{V_v}) \quad \text{and} \quad \rho_\varphi(\mathcal{F}_{V_i}) = \mathcal{F}_{V_v}. \\ &\mu^{-1}_{wy_v} \int_{V_v} |\rho_\varphi u|^2 d\mu = \mu^{-1}_{w_i} \int_{V_i} |u|^2 d\mu, \qquad u \in L^2(V_i, \mu|_{V_i}). \\ &r_{wy_v} \mathcal{E}(\rho_\varphi u, \rho_\varphi u) = r_{w_i} \mathcal{E}(u,u), \qquad u \in \mathcal{F}_{V_i}. \end{aligned} \tag{7.8}$$

Since $\varphi|_{\Lambda^1_{s_v,x_v}} : \Lambda^1_{s_v,x_v} \to \Lambda^1_{s_i,x_i}$ is an $\mathcal{L}$-isomorphism and $F_\varphi|_{K_{wy_v}} = F_{w_i} \circ F^{-1}_{wy_v}$, it follows that $F_\varphi\bigl(U_{s_v}(x_v, \mathcal{S})\bigr) = U_{s_i}(x_i, \mathcal{S})$ and that $F_\varphi\bigl(V_v \setminus K_{wy_v}[X]\bigr) = V_i \setminus K_{w_i}[X]$. Therefore

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(7.8) together with (CHK) of $(\mathcal{E}^U, \mathcal{F}_U)$ for a non-empty open subset U of K implies that for any $(s,x,y) \in (0,\infty)\times V_v \times V_v$,

$$\begin{aligned} \mu_{wy_v} p^{V_v}_{ss_v^2}(x,y) &= \mu_{w_i} p^{V_i}_{s\gamma^2_{w_i}}(F_\varphi(x), F_\varphi(y)), \\ \mu_{wy_v} p^{V_v \setminus K_{wy_v}[X]}_{ss_v^2}(x,y) &= \mu_{w_i} p^{V_i \setminus K_{w_i}[X]}_{s\gamma^2_{w_i}}(F_\varphi(x), F_\varphi(y)). \end{aligned} \tag{7.9}$$

Since $wy_v \le v$ and $s_v = \gamma_{wy_v} \le \sqrt{t}$ we have $U_v = K\bigl(\Lambda_{s_v, wy_v}\bigr) \subset K\bigl(\Lambda_{\sqrt{t},v}\bigr)$. Therefore Lemma 7.12 and (7.9) imply that

$$\begin{aligned} &\int_{K(\Lambda_{\sqrt{t},v})} \bigl(p^N_t(x,x) - p^{K\setminus K_w[X]}_t(x,x)\bigr)\, d\mu(x) \\ &\ge \int_{U_v} \bigl(p^N_t(x,x) - p^{K\setminus K_{wy_v}[X]}_t(x,x)\bigr)\, d\mu(x) \ge \int_{U_v} \bigl(p^{V_v}_t(x,x) - p^{V_v\setminus K_{wy_v}[X]}_t(x,x)\bigr)\, d\mu(x) \\ &= \mu_{w_i}\mu^{-1}_{wy_v} \int_{U_v} \bigl(p^{V_i}_{\gamma^2_{w_i} s_v^{-2} t}(F_\varphi(x), F_\varphi(x)) - p^{V_i\setminus K_{w_i}[X]}_{\gamma^2_{w_i} s_v^{-2} t}(F_\varphi(x), F_\varphi(x))\bigr)\, d\mu(x) \\ &= \int_{U_i} \bigl(p^{V_i}_{\gamma^2_{w_i} s_v^{-2} t}(x,x) - p^{V_i\setminus K_{w_i}[X]}_{\gamma^2_{w_i} s_v^{-2} t}(x,x)\bigr)\, d\mu(x). \end{aligned}$$

Recall that $1 \le s_v^{-2} t \le \beta_X$, where $\beta_X := \gamma^{-2(M_X+1)}$, hence $\gamma^2_{w_i} s_v^{-2} t \in [\gamma^2_{w_i}, \beta_X \gamma^2_{w_i}]$. Therefore for the proof of Lemma 7.13 it suffices to prove that for any $a, b \in (0,\infty)$ with $a < b$ and for any $i \in \{1,\dots,J\}$ satisfying $\Lambda_{s_i,x_i} = \{w_i\}$,

$$\inf_{t\in[a,b]} \int_{U_i} \bigl(p^{V_i}_t(x,x) - p^{V_i\setminus K_{w_i}[X]}_t(x,x)\bigr)\, d\mu(x) > 0. \tag{7.10}$$

Let $a, b \in (0,\infty)$, $a < b$ and let $i \in \{1,\dots,J\}$ satisfy $\Lambda_{s_i,x_i} = \{w_i\}$. Since K is assumed to be connected, it is also arcwise connected by [27, Theorem 1.6.2], and any non-empty open subset of K is locally arcwise connected. Let $V_{c,i}$ be the connected component of V_i containing x_i. Then $V_{c,i}$ is an arcwise connected open subset of K. By Lemma 7.11 (2) and Proposition A.3 (2), $p^{V_i}_t(x,y) \ge p^{V_{c,i}}_t(x,y) > 0$ for any $(t,x,y) \in (0,\infty)\times V_{c,i}\times V_{c,i}$. On the other hand, we have $U_i \subset V_{c,i}$ since U_i is connected and $x_i \in K_{w_i} \subset U_i \subset V_i$. Hence by (CHK) of $(\mathcal{E}^{V_i}, \mathcal{F}_{V_i})$,

$$q_i := \inf\bigl\{p^{V_i}_t(x,y) \mid (t,x,y) \in [a/2, b]\times U_i \times U_i\bigr\} > 0. \tag{7.11}$$

We write $V_{\Delta,i} := (V_i)_\Delta$ and define $\sigma^i_A(\omega) := \inf\bigl\{t \in [0,\infty) \mid X^{V_i}_t(\omega) \in A\bigr\}$ ($\inf\emptyset := \infty$) and $\tau^i_A(\omega) := \sigma_{V_{\Delta,i}\setminus A}(\omega)$ for $A \in \mathcal{B}(V_{\Delta,i})$ and $\omega \in \Omega$. $\mathrm{Cap}_{\mathcal{E}}(K[X]) > 0$ and Lemma 7.14 imply $\mathrm{Cap}_{\mathcal{E}}\bigl(K_{w_i}[X]\bigr) > 0$. Since $K_{w_i}[X] \subset V_i$ and $K_{w_i}[X]$ is compact, [17, Theorem 4.4.3 (ii)] implies that $\mathrm{Cap}_{\mathcal{E}^{V_i}}\bigl(K_{w_i}[X]\bigr) \in (0,\infty)$. Therefore by Theorem A.5, there exist a $\mu|_{V_i}$-regular closed subset F_i of V_i, $z_i \in K_{w_i}[X] \cap F_i$ and an open neighborhood G_i of z_i in V_i such that

$$h_i := \inf_{x\in G_i\cap F_i} \mathbf{P}_x\bigl[\sigma^i_{K_{w_i}[X]} \le a/2\bigr] > 0. \tag{7.12}$$

Let $A_i := F_i \cap G_i \cap \mathrm{int}_K U_i$. Note that $G_i \cap \mathrm{int}_K U_i$ is an open neighborhood of z_i in V_i since $z_i \in K_{w_i}[X] \subset K_{w_i} \subset \mathrm{int}_K U_i$. Therefore $\mu(A_i) > 0$ by the $\mu|_{V_i}$-regularity of F_i.

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Now let $t \in [a,b]$ and $x \in A_i$. Since $(\mathcal{E}^U, \mathcal{F}_U)$ satisfies (CHK) for $U = V_i \setminus K_{w_i}[X], V_i$,

$$\begin{aligned} p_t^{V_i}(x,x) - p_t^{V_i\setminus K_{w_i}[X]}(x,x) &= \lim_{s\downarrow 0} \frac{\int_{U_s(x,\mathcal{S})} \big(p_t^{V_i}(x,y) - p_t^{V_i\setminus K_{w_i}[X]}(x,y)\big) d\mu(y)}{\mu(U_s(x,\mathcal{S}))} \\ &= \lim_{s\downarrow 0} \frac{\mathbf{P}_x\big[X_t^{V_i} \in U_s(x,\mathcal{S})\big] - \mathbf{P}_x\big[X_t^{V_i} \in U_s(x,\mathcal{S}), t < \sigma^i_{K_{w_i}[X]}\big]}{\mu(U_s(x,\mathcal{S}))} \\ &= \lim_{s\downarrow 0} \frac{\mathbf{P}_x\big[X_t^{V_i} \in U_s(x,\mathcal{S}), \sigma^i_{K_{w_i}[X]} \le t\big]}{\mu(U_s(x,\mathcal{S}))}. \end{aligned} \tag{7.13}$$

As $x \in \mathrm{int}_K U_i$, we may choose $\delta \in (0,1]$ so that $U_s(x,\mathcal{S}) \subset \mathrm{int}_K U_i$ for any $s \in (0,\delta]$. Let $s \in (0,\delta]$. In the calculation below, we write $X^i(t,\omega) := X_t^{V_i}(\omega)$ for each $(t,\omega) \in [0,\infty]\times\Omega$ and $\sigma_i := \sigma^i_{K_{w_i}[X]}$. Since $X^i(\sigma_i(\omega),\omega) \in K_{w_i}[X]$ for $\omega \in \{\sigma_i < \infty\}$, by the strong Markov property of the diffusion X^{V_i} (see [26, Corollary 2.6.18], for example), (7.11) and (7.12),

$$\begin{aligned} &\mathbf{P}_x\big[X_t^{V_i} \in U_s(x,\mathcal{S}), \sigma_i \le t\big] \ge \mathbf{P}_x\big[X_t^{V_i} \in U_s(x,\mathcal{S}), \sigma_i \le a/2\big] \\ &= \int_{\{\sigma_i \le a/2\}} \mathbf{P}_{X^i(\sigma_i(\omega),\omega)}\big[X^{V_i}_{t-\sigma_i(\omega)} \in U_s(x,\mathcal{S})\big] d\mathbf{P}_x(\omega) \\ &= \int_{\{\sigma_i \le a/2\}} \left[\int_{U_s(x,\mathcal{S})} p^{V_i}_{t-\sigma_i(\omega)}\big(X^i(\sigma_i(\omega),\omega), y\big) d\mu(y)\right] d\mathbf{P}_x(\omega) \\ &\ge \int_{\{\sigma_i \le a/2\}} \left[\int_{U_s(x,\mathcal{S})} q_i d\mu(y)\right] d\mathbf{P}_x(\omega) = q_i \mathbf{P}_x[\sigma_i \le a/2] \mu(U_s(x,\mathcal{S})) \ge q_i h_i \mu(U_s(x,\mathcal{S})). \end{aligned}$$

Hence the limit in (7.13) is bounded from below by $q_i h_i$, that is,

$$p_t^{V_i}(x,x) - p_t^{V_i\setminus K_{w_i}[X]}(x,x) \ge q_i h_i, \qquad (t,x) \in [a,b]\times A_i. \tag{7.14}$$

Therefore for any $t \in [a,b]$, since $A_i \subset U_i$,

$$\begin{aligned} \int_{U_i} \big(p_t^{V_i}(x,x) - p_t^{V_i\setminus K_{w_i}[X]}(x,x)\big) d\mu(x) &\ge \int_{A_i} \big(p_t^{V_i}(x,x) - p_t^{V_i\setminus K_{w_i}[X]}(x,x)\big) d\mu(x) \\ &\ge q_i h_i \mu(A_i)\,(> 0), \end{aligned}$$

proving (7.10). This completes the proofs of Lemma 7.13 and of Theorem 7.7. $\blacksquare_{\textbf{Lemma 7.13}}$

Proof of Corollary 7.8. As $\Sigma_{w_J}[X_J] \subset \mathcal{P}_{\mathcal{L}}$ for some $w_J \in W_*$ by (RB), we also have $\Sigma[X_J] \subset \mathcal{P}_{\mathcal{L}}$ and hence $K[X_J] \subset V_0$. By $K \ne V_0\ (= \overline{V_0})$ we may choose $w \in W_\#$ so that $K_w \cap V_0 = \emptyset$. Let $\ell := |w|$ and $V_\ell := \bigcup_{v\in W_\ell} F_v(V_0)$. Then $(\emptyset \ne) K^I_{W_\ell} = V_\ell^c \subset K^I = V_0^c$ by Lemma 2.11, hence $V_0 \subset V_\ell \subsetneqq K$. Since $(\mathcal{L},\mathcal{S})$ is locally finite, Proposition 6.7 (1) implies that $\dim_{\mathcal{S}} V_\ell = d_\partial = \dim_{\mathcal{S}} K_w[X_J]$. Therefore $\dim_{\mathcal{S}}(V_\ell \setminus V_0) = d_\partial$ by $K_w[X] \subset V_\ell \setminus V_0$. By Theorem 7.7 there exists $c_3 \in (0,\infty)$ such that $Z_{K^I}(t) - Z_{K^I\setminus K_w[X_J]}(t) \ge c_3 t^{-d_\partial/2}$ for any $t \in (0,1]$. Also Theorem 5.11 implies that there exists $c_4 \in (0,\infty)$ such that $0 \le Z_{K^I}(t) - Z_{K^I_{W_\ell}}(t) \le c_4 t^{-d_\partial/2}$ for any $t \in (0,1]$. Note that $Z_{K^I_{W_\ell}} = Z_{V_\ell^c} \le Z_{K^I\setminus K_w[X_J]}$, that $Z_{K^I} = Z_D$ and that $Z_{K^I_{W_\ell}}(t) = (\#S)^\ell Z_D(t\gamma^{-2\ell}) \le Z_D(t)$ for any $t \in (0,\infty)$ by Lemma 5.7. Hence we conclude that

$$c_3 t^{-d_\partial/2} \le Z_D(t) - (\#S)^\ell Z_D\Big(\frac{t}{\gamma^{2\ell}}\Big) \le c_4 t^{-d_\partial/2}, \qquad t \in (0,1]. \tag{7.15}$$

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Since $\mathrm{Cap}_{\mathcal{E}}(V_0) > 0$ by $V_0 \supset K[X_J]$ and $\mathrm{Cap}_{\mathcal{E}}(K[X_J]) > 0$ (or by Theorem 7.18, which is presented in the next section), [17, Corollary 2.3.1] implies that $\boldsymbol{1} \notin \mathcal{F}_{K^I}$. By [27, Theorem 1.6.2], Lemma 7.11 (1) and Proposition A.3 (2), $(\mathcal{E}, \mathcal{F})$ is irreducible. [12, Theorem 2.1.11] implies that $\mathcal{E}(u,u) > 0$ for any $u \in \mathcal{F}_{K^I} \setminus \{0\}$. Hence $\lambda_1^D := \lambda_1(K^I) > 0$ (recall (4.12)). Let $Z_D^1(t) := e^{\lambda_1^D t} Z_D(t)$, $t \in (0,\infty)$. Then Z_D^1 is clearly $(0,\infty)$-valued and non-increasing. Therefore

$$0 \le Z_D(t) - (\#S)^\ell Z_D\Bigl(\frac{t}{\gamma^{2\ell}}\Bigr) \le Z_D(t) = Z_D^1(t) e^{-\lambda_1^D t} \le Z_D^1(1) e^{-\lambda_1^D t}, \quad t \in [1,\infty). \quad (7.16)$$

Now by (7.15) and (7.16), we can follow the arguments of [27, Proofs of Theorems 4.1.5 and B.4.3] to prove that there exists a continuous $\log(\gamma^{-\ell})$-periodic function $G_\ell : \mathbb{R} \to (0,\infty)$ and $c_1, c_2 \in (0,\infty)$ such that (7.2) holds for any $t \in (0,1]$, with G replaced by G_ℓ. But then $G_\ell = G$ since G_ℓ and G are both $\log(\gamma^{-\ell})$-periodic. ■**Corollary 7.8**

7.3. Positivity of capacity for subsets of the boundary

As in the previous subsection, in this section $(\mathcal{L} = (K, S, \{F_i\}_{i\in S}), \mu, \mathcal{E}, \mathcal{F}, \mathbf{r} = (r_i)_{i\in S})$ is assumed to be a self-similar Dirichlet space with μ a self-similar measure with weight $(\mu_i)_{i\in S}$ and $\mathcal{S} = \{\Lambda_s\}_{s\in(0,1]}$ to be the self-similar scale with weight $\boldsymbol{\gamma} := (\gamma_i)_{i\in S}$, $\gamma_i := \sqrt{r_i \mu_i}$. As usual, let $\pi : \Sigma \to K$ denote the canonical projection.

The purpose of this subsection is to state and prove the following Theorem 7.18, which asserts that every subset of $\overline{V_0}$ with non-empty interior in $\overline{V_0}$ has positive capacity. This kind of statement is indispensable when we apply Theorem 7.7 to concrete examples.

Notation. For each $u \in \mathcal{F}$, its quasi-continuous modification, which exists and is unique up to $\mathcal{E}$-q.e., is denoted by $\tilde{u}$. Note that $\mathcal{F}_U = \{u \in \mathcal{F} \mid \tilde{u} = 0\ \mathcal{E}\text{-q.e. on } K \setminus U\}$ for any non-empty open subset U of K by [17, Corollary 2.3.1]. See [17, Chapter 2] for details.

Theorem 7.18 *Assume that K is connected, that $(\mathcal{E}, \mathcal{F})$ is conservative and that* (CHK) *and* (UHK) *hold. Then $\mathrm{Cap}_{\mathcal{E}}(G) > 0$ for any non-empty open subset G of $\overline{V_0}$.*

Remark. Since $\#S \ge 2$, the connectivity of K implies that $V_0 \ne \emptyset$.

Proof. Let G be a non-empty open subset of $\overline{V_0}$. Then we may choose an open subset O of K so that $G = O \cap \overline{V_0}$. Also there exists $x \in O \cap V_0$. Let $\omega \in \pi^{-1}(x)$. Then $\omega \in \mathcal{P}_{\mathcal{L}}$ since $\pi^{-1}(V_0) = \mathcal{P}_{\mathcal{L}}$ by [27, Proposition 1.3.5 (1)]. Therefore there exist $w = w_1 \dots w_m \in W_\#$ and $j \in S \setminus \{w_1\}$ such that $F_w(x) \in K_w \cap K_j$ (recall Definition 2.10 (2)). Since $F_w : K \to K_w$ is a homeomorphism, $F_w(O)$ is an open subset of K_w and we can choose an open subset O_w of K so that $F_w(O) = O_w \cap K_w$. Then $F_w(x) \in F_w(O \cap V_0) = O_w \cap F_w(V_0)$. Let U_w be the connected component of O_w containing $F_w(x)$ and set $U := F_w^{-1}(U_w) (\ni x)$. Then $F_w(x) \in U_w \cap K_w \cap K_j \ne \emptyset$, and as in the proof of (7.10), U_w is an arcwise connented open subset of K. Also, $F_w(U \cap \overline{V_0}) = U_w \cap F_w(\overline{V_0}) \subset O_w \cap F_w(\overline{V_0}) = F_w(O) \cap F_w(\overline{V_0}) = F_w(G)$ and therefore $U \cap \overline{V_0} \subset G$. Since $\mathrm{int}_K \overline{V_0} = \emptyset$ by [28, Theorem 1.2.2], $U \cap K^I = U \setminus \overline{V_0} \ne \emptyset$ and $U_w \cap K_w^I = F_w(U \cap K^I) \ne \emptyset$. Similarly, since $F_j^{-1}(U_w)$ is also a non-empty open subset of K, $F_j^{-1}(U_w) \cap K^I = F_j^{-1}(U_w) \setminus \overline{V_0} \ne \emptyset$ and $U_w \cap K_j^I = F_j\bigl(F_j^{-1}(U_w) \cap K^I\bigr) \ne \emptyset$.

Now suppose $\mathrm{Cap}_{\mathcal{E}}(G) = 0$. Then $\mathrm{Cap}_{\mathcal{E}}(U \setminus (U \cap K^I)) = \mathrm{Cap}_{\mathcal{E}}(U \cap \overline{V_0}) = 0$ and therefore $\mathcal{F}_U = \mathcal{F}_{U \cap K^I}$. Let $u \in \mathcal{F}_{U_w}$. Then $\tilde{u} = 0$ $\mathcal{E}$-q.e. on $K \setminus U_w$. Using Lemma 7.14,

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we see that $\tilde{u}\circ F_w$ is a quasi-continuous modification of $u\circ F_w \in \mathcal{F}$. As $F_w(\{y \in K \setminus U \mid \tilde{u}\circ F_w(y) \neq 0\}) \subset \{y \in K \setminus U_w \mid \tilde{u}(y) \neq 0\}$, Lemma 7.14 yields

$$\begin{aligned} &\min\{r_w^{-1}, \mu_w\}\mathrm{Cap}_{\mathcal{E}}\bigl(\{y \in K \setminus U \mid \tilde{u}\circ F_w(y) \neq 0\}\bigr) \\ &\leq \mathrm{Cap}_{\mathcal{E}}\bigl(F_w(\{y \in K \setminus U \mid \tilde{u}\circ F_w(y) \neq 0\})\bigr) \leq \mathrm{Cap}_{\mathcal{E}}\bigl(\{y \in K \setminus U_w \mid \tilde{u}(y) \neq 0\}\bigr) = 0. \end{aligned}$$

Therefore $\tilde{u}\circ F_w = 0$ $\mathcal{E}$-q.e. on $K \setminus U$, hence $u\circ F_w \in \mathcal{F}_U = \mathcal{F}_{U\cap K^I} \subset \mathcal{F}_{K^I}$. By Lemma 5.5, $u_w := u\cdot\boldsymbol{1}_{K_w} = (u\circ F_w)^w \in \mathcal{F}_{K^I_w}$, which implies $\widetilde{u_w} = 0$ $\mathcal{E}$-q.e. on $K \setminus K^I_w$. Since $\widetilde{u_w} = u\cdot\boldsymbol{1}_{K_w} = u = \tilde{u}$ μ-a.e. on K^I_w, [17, Lemma 2.1.4] yields $\widetilde{u_w} = \tilde{u}$ $\mathcal{E}$-q.e. on K^I_w. Also, $\tilde{u} = 0$ $\mathcal{E}$-q.e. on $K \setminus U_w$ by $u \in \mathcal{F}_{U_w}$ and therefore $\widetilde{u_w} = \tilde{u} = 0$ $\mathcal{E}$-q.e. on $K^I_w \setminus U_w$. Thus $\widetilde{u_w} = 0$ $\mathcal{E}$-q.e. on $K \setminus (U_w \cap K^I_w)$, hence $u\cdot\boldsymbol{1}_{U_w\cap K_w} = u\cdot\boldsymbol{1}_{K_w} = u_w \in \mathcal{F}_{U_w\cap K^I_w} (\subset \mathcal{F}_{U_w})$ and $u\cdot\boldsymbol{1}_{U_w\setminus K_w} = u - u\cdot\boldsymbol{1}_{U_w\cap K_w} \in \mathcal{F}_{U_w}$. Recalling $|w| = m$, it follows that, for any $u, v \in \mathcal{F}_{U_w}$,

$$\mathcal{E}(u\cdot\boldsymbol{1}_{U_w\cap K_w}, v\cdot\boldsymbol{1}_{U_w\setminus K_w}) = \sum_{\tau\in W_m} \frac{1}{r_\tau}\mathcal{E}\bigl((u\cdot\boldsymbol{1}_{U_w\cap K_w})\circ F_\tau, (v\cdot\boldsymbol{1}_{U_w\setminus K_w})\circ F_\tau\bigr) = 0,$$

$$\text{hence}\qquad \mathcal{E}(u, v) = \mathcal{E}(u\cdot\boldsymbol{1}_{U_w\cap K_w}, v\cdot\boldsymbol{1}_{U_w\cap K_w}) + \mathcal{E}(u\cdot\boldsymbol{1}_{U_w\setminus K_w}, v\cdot\boldsymbol{1}_{U_w\setminus K_w}). \quad (7.17)$$

Since $\mu(U_w \cap K_w) \geq \mu(U_w \cap K^I_w) > 0$ and $\mu(U_w \setminus K_w) \geq \mu(U_w \cap K^I_j) > 0$, (7.17) together with [17, Theorem 1.6.1] contradicts the fact that $(\mathcal{E}^{U_w}, \mathcal{F}_{U_w})$ is irreducible by Proposition A.3 (2) and the arcwise connectivity of U_w. Thus $\mathrm{Cap}_{\mathcal{E}}(G) > 0$ follows. ■

8. Examples: Sierpinski carpets

In this section, we illustrate the results of the previous sections by applying them to a class of infinitely ramified self-similar sets called *generalized Sierpinski carpets*, whose definition was originally given by Barlow and Bass [6, Section 2] but has recently been modified by Barlow, Bass, Kumagai and Teplyaev [7], Hino [23] and Kigami [28, Section 3.4]. We follow the formulation of Hino [23] in the argument below, but their formulations of generalized Sierpinski carpets are all equivalent, as stated in Kajino [25, Section 2].

Definition 8.1 (Generalized Sierpinski carpets) Let $d \in \mathbb{N}$ and set $Q_0 := [0,1]^d$. Let $L \in \mathbb{N}$, $L \geq 2$ and set $\mathcal{Q}_1 := \bigl\{\prod_{i=1}^d \bigl[(k_i - 1)L^{-1}, k_i L^{-1}\bigr] \mid k_1, \ldots, k_d \in \{1, \ldots, L\}\bigr\}$. Let $S \subset \mathcal{Q}_1$ be non-empty, and for each $q \in S$ we define $F_q : \mathbb{R}^d \to \mathbb{R}^d$ by $F_q(x) := L^{-1}x + z^q$, where $z^q \in \mathbb{R}^d$ is chosen so that $F_q(Q_0) = q\,(\subset Q_0)$. We also set $Q^S_1 := \bigcup_{q\in S} q$.

Let $\mathrm{GSC}(d, L, S)$ be the self-similar set associated with $\{F_q\}_{q\in S}$, that is, the unique non-empty compact subset K of $\mathbb{R}^d$ that satisfies $K = \bigcup_{q\in S} F_q(K)$. We call $\mathrm{GSC}(d, L, S)$ a *generalized Sierpinski carpet* if and only if S satisfies the following four conditions:
(GSC1) (Symmetry) Q^S_1 is preserved by all the isometries of Q_0.
(GSC2) (Connectedness) Q^S_1 is connected.
(GSC3) (Non-diagonality) If B is a d-dimensional rectangle with each side length L^{-1} or $2L^{-1}$ which is the union of elements of $\mathcal{Q}_1$, $\mathrm{int}_{\mathbb{R}^d}(B \cap Q^S_1)$ is either empty or connected.
(GSC4) (Borders included) $\{(x_1, 0, \ldots, 0) \mid x_1 \in [0,1]\} \subset Q^S_1$.

In particular, we call $\mathrm{GSC}(2, 3, S_{\mathrm{SC}})$ the *Sierpinski carpet* (see Figure 1.2), where $S_{\mathrm{SC}} := \{[(k_1 - 1)/3, k_1/3] \times [(k_2 - 1)/3, k_2/3] \mid (k_1, k_2) \in \{1, 2, 3\}^2 \setminus \{(2, 2)\}\}$.

In the rest of this section, we fix a generalized Sierpinski carpet $\mathrm{GSC}(d, L, S)$. Let $K := \mathrm{GSC}(d, L, S)$ and $\mathcal{L} := (K, S, \{F_q\}_{q\in S})$ be the self-similar structure associated with $\{F_q\}_{q\in S}$. The following proposition is immediate by the assumptions.

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Proposition 8.2 *(1) K is connected (by* (GSCj)*, $j=1,2,4$ and [27, Theorem 1.6.2]).*
(2) Let $k \in \{1,\dots,d\}$. Set $H_{k,s} := \{(x_1,\dots,x_d) \in \mathbb{R}^d \mid x_k = s\}$ and $S_{k,s} := \{q \in S \mid q \cap H_{k,s} \neq \emptyset\}$ for $s \in [0,1]$ and let $R_k : \mathbb{R}^d \to \mathbb{R}^d$ be the reflection in the hyperplane $H_{k,1/2}$. Then R_k induces natural bijections $S_{k,0} \to S_{k,1}$ and $S_{k,1} \to S_{k,0}$ given by $q \mapsto R_k(q)$.
(3) $\mathcal{L}$ satisfies (RB) *with $\mathcal{P}_{\mathcal{L}} = \bigcup_{k=1}^{d}(\Sigma[S_{k,0}] \cup \Sigma[S_{k,1}])$, and $V_0 = K \setminus (0,1)^d \neq K$.*

Next we discuss self-similar Dirichlet forms on K. We follow the arguments in [28, Section 3.4]. Let ν be the self-similar measure on K with weight $((\#S)^{-1},\dots,(\#S)^{-1})$. By combining the arguments of Barlow and Bass [5, 6] and Kusuoka and Zhou [32], as in Hambly, Kumagai, Kusuoka and Zhou [22] (note also the recent result [7] on uniqueness of the Dirichlet form on generalized Sierpinski carpets), we have a conservative self-similar Dirichlet space (in the sense of Definition 3.3) $(\mathcal{L},\nu,\mathcal{E},\mathcal{F},\mathbf{r}=(r_q)_{q\in S})$ satisfying (SSDF3$\mathcal{S}$), (CHK) and (UHK) and with $r_q = r$ for any $q \in S$ for some $r \in (0,\infty)$.

Now let μ be a self-similar measure on K with weight $(\mu_q)_{q\in S}$ satisfying $r\mu_q < 1$ for any $q \in S$. By a result of Barlow and Kumagai [11, Lemma 2.5], μ is *smooth with respect to* $(\mathcal{E},\mathcal{F})$, that is, $\mu(A) = 0$ for any $A \in \mathcal{B}(K)$ with $\mathrm{Cap}_{\mathcal{E}}(A) = 0$. By [17, Theorem 6.2.1], we can construct the *time changed Dirichlet space* $(\mathcal{E}^\mu,\mathcal{F}_\mu)$ *of* $(\mathcal{E},\mathcal{F})$ *with respect to* μ, which is a regular Dirichlet form on $L^2(K,\mu)$. Since the whole space K is a quasi-support of μ by [11, Proposition 2.6] and [17, (5.1.22) and Theorem 5.1.5], [17, Theorem 1.5.2 (iii) and (6.2.22)] yield $\mathcal{F}_\mu \cap C(K) = \mathcal{F} \cap C(K)$ and $\mathcal{E}^\mu(u,v) = \mathcal{E}(u,v)$ for any $u,v \in \mathcal{F} \cap C(K)$. Therefore $(\mathcal{L},\mu,\mathcal{E}^\mu,\mathcal{F}_\mu,\mathbf{r})$ is a conservative self-similar Dirichlet space satisfying (SSDF3$\mathcal{S}$). Moreover, by the discussions of [11] (see also [28, Section 3.4]), we can verify (CHK) and the assumptions of [28, Theorem 3.2.3] for $(\mathcal{L},\mu,\mathcal{E}^\mu,\mathcal{F}_\mu,\mathbf{r})$. Finally, let $\gamma^\mu_q := \sqrt{r\mu_q}$ for $q \in S$, $\boldsymbol{\gamma}^\mu := (\gamma^\mu_q)_{q\in S}$ and $\mathcal{S}^\mu = \{\Lambda^\mu_s\}_{s\in(0,1]}$ be the self-similar scale with weight $\boldsymbol{\gamma}^\mu$. Then by [28, Theorems 3.2.3, 3.4.5 and Proof of Lemma 3.5.16], we have the following criterion for (UHK), (LWTF) and $(\mathcal{L},\mathcal{S}^\mu)$ being intersection type finite (see also [25, Proposition 3.3 and Theorem 3.5] for a short self-contained treatment of (VD)).

Proposition 8.3 *The following four conditions are equivalent.*
(0) $(\mu_q)_{q\in S}$ is weakly symmetric, i.e. $\mu_q = \mu_{R_k(q)}$ for any $k \in \{1,\dots,d\}$ and any $q \in S_{k,0}$.
(1) $(\mathcal{L},\mathcal{S}^\mu)$ is locally finite.
(2) $(\mathcal{L},\mathcal{S}^\mu,\mu)$ satisfies (VD)*.*
(3) (UHK) *holds for $(\mathcal{L},\mu,\mathcal{E}^\mu,\mathcal{F}_\mu,\mathbf{r})$.*
Moreover, if any one of these four conditions holds, then $(\mathcal{L},\mathcal{S}^\mu)$ is intersection type finite and $(\mathcal{L},\mu,\mathcal{E}^\mu,\mathcal{F}_\mu,\mathbf{r})$ satisfies (LWTF)*.*

Hence we conclude that *if $(\mu_q)_{q\in S}$ is weakly symmetric then all the statements of Theorem 5.2 are valid for $(\mathcal{L},\mu,\mathcal{E}^\mu,\mathcal{F}_\mu,\mathbf{r})$* with $d_\partial = d^\mu_\partial := \max\{d(\boldsymbol{\gamma}^\mu,S_{k,0}) \mid k \in \{1,\dots,d\}\}(= \dim_{\mathcal{S}^\mu} V_0 < \dim_{\mathcal{S}^\mu} K$ in view of Theorem 6.9).

Moreover, suppose that $(\mu_q)_{q\in S}$ is weakly symmetric. Then Theorem 7.18 implies that $\mathrm{Cap}_{\mathcal{E}^\mu}(K[S_{k,j}]) > 0$ for any $k \in \{1,\dots,d\}$ and any $j \in \{0,1\}$. Therefore Theorem 7.7 implies the following reminder estimate. For $U \subset K$ non-empty open, let $Z_{U,\mu}$ denote the partition function associated with $\bigl((\mathcal{E}^\mu)^U,(\mathcal{F}_\mu)_U\bigr)$, $Z_{N,\mu} := Z_{K,\mu}$ and $Z_{D,\mu} := Z_{K^I,\mu}$.

Theorem 8.4 *Assume that $(\mu_q)_{q\in S}$ is weakly symmetric. Let $k \in \{1,\dots,d\}$, $j \in \{0,1\}$ and $d^\mu_k := d(\boldsymbol{\gamma}^\mu,S_{k,0})$. Then there exist $c_1,c_2 \in (0,\infty)$ such that for any $t \in (0,1]$,*

$$c_1 t^{-d^\mu_k/2} \le Z_{N,\mu}(t) - Z_{K\setminus K[S_{k,j}],\mu}(t) \le c_2 t^{-d^\mu_k/2}. \tag{8.1}$$

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On the other hand, if $\mu_q = (\#S)^{-1}$ for any $q \in S$, i.e. $\mu = \nu$, then ${d_{\mathrm{Euc}}}^{d_w/2}$ is a $(2/d_w)$-qdistance adapted to $\mathcal{S}^\nu$, where d_{Euc} is the Euclidean distance and $d_w := \log_L(\#S/r)$. Hence by Proposition 2.24, $\dim_{\mathcal{S}^\nu} K = 2d_f/d_w$ and $\dim_{\mathcal{S}^\nu} V_0 = 2d_b/d_w$ in this case, where $d_f := \log_L(\#S)$ and $d_b := \log_L(\#S_{1,0})$. Therefore Corollary 7.8 implies the following sharp reminder estimate for $Z_{D,\nu}$.

Theorem 8.5 *Let G be the $\log(\#S/r)^{1/2}$-periodic function given by Corollary 5.3 for $(\mathcal{L}, \nu, \mathcal{E}, \mathcal{F}, \mathbf{r})$. Then there exist $c_3, c_4 \in (0, \infty)$ such that for any $t \in (0, 1]$,*

$$c_3 t^{-d_b/d_w} \le t^{-d_f/d_w} G\Big(\frac{1}{2}\log\frac{1}{t}\Big) - Z_{D,\nu}(t) \le c_4 t^{-d_b/d_w}. \tag{8.2}$$

9. Concluding remarks

We conclude the present paper with a brief discussion of open problems.

Consider the situation of Theorem 5.2. In the non-lattice case, we have shown an asymptotic behavior of the eigenvalue counting functions (Corollary 5.4) by virtue of Karamata's Tauberian theorem. Unfortunately, in the lattice case we do not have any similar result for the eigenvalue counting functions. The main difficulty here is that the T-periodic function G given in Theorem 5.2 may be non-constant. In this case, it seems hopeless to verify the so-called *'Tauberian conditions'* on G.

It also seems extremely difficult to apply the renewal theorem directly to the eigenvalue counting function, since we cannot use probabilistic arguments to estimate $N_N(x) - N_D(x)$. This is why Hambly [21] and this article have treated the partition function mainly and not the eigenvalue counting function.

A. Appendix — Miscellaneous lemmas for Section 7

In this appendix, we present basic results on continuity and positivity of heat kernels and positivity of hitting probabilities for regular Dirichlet forms. Those results play essential roles in the proof of Theorem 7.7. Let E be a locally compact separable metrizable space and let $E_\Delta := E \cup \{\Delta_E\}$ denote its one-point compactification. Throughout this appendix, we assume that μ is a Borel measure on E satisfying $\mu(F) < \infty$ for any compact $F \subset E$ and $\mu(O) > 0$ for any non-empty open $O \subset E$, that $(\mathcal{E}, \mathcal{F})$ is a (symmetric) regular Dirichlet form on $L^2(E, \mu)$ and that H and $\{T_t\}_{t\in(0,\infty)}$ are the non-negative self-adjoint operator with domain $\mathcal{D}[H]$ and the strongly continuous contraction semigroup, respectively, associated with the closed form $(\mathcal{E}, \mathcal{F})$ on $L^2(E, \mu)$.

The following definition is just a reminder for the readers.

Definition A.1 (1) $\{T_t\}_{t\in(0,\infty)}$ is called *ultracontractive* if and only if $T_t(L^2(E, \mu)) \subset L^\infty(E, \mu)$ and $T_t : L^2(E, \mu) \to L^\infty(E, \mu)$ is a bounded linear operator for any $t \in (0, \infty)$.
(2) A family $\{p_t\}_{t\in(0,\infty)}$ of $\mathbb{R}$-valued $\mathcal{B}(E\times E)$-measurable functions on $E\times E$ is called a *heat kernel of* $\{T_t\}_{t\in(0,\infty)}$ if and only if for each $t \in (0, \infty)$ and for any $f \in L^2(E, \mu)$,

$$T_t f = \int_E p_t(\cdot, y) f(y) d\mu(y) \quad \mu\text{-a.e. on } E. \tag{A.1}$$

Clearly, for $t \in (0, \infty)$, such an integral kernel p_t of T_t, if exists, is unique up to $\mu\times\mu$-a.e. and satisfies $p_t(x, y) = p_t(y, x) \ge 0$ $\mu\times\mu$-a.e. on $K\times K$. See [20, Section 2] for details.

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(3) We say that $(\mathcal{E},\mathcal{F})$ satisfies (CHK), or simply (CHK) holds, if and only if $\{T_t\}_{t\in(0,\infty)}$ admits a heat kernel $\{p_t\}_{t\in(0,\infty)}$ which is jointly continuous, i.e. such that $p = p_t(x,y) : (0,\infty)\times E\times E \to \mathbb{R}$ is continuous. Clearly, such $\{p_t\}_{t\in(0,\infty)}$, if exists, is unique.

By [14, Theorem 2.1.4], if $\mu(E) < \infty$ then the ultracontractivity of $\{T_t\}_{t\in(0,\infty)}$ implies the existence of a heat kernel $\{p_t\}_{t\in(0,\infty)} \subset L^\infty(E\times E, \mu\times\mu)$.

Next let us recall the following definitions. See [17, Section 1.4] for the definitions of (sub-)Markovian transition functions, their μ-symmetry and the Markovian semigroup on $L^2(E,\mu)$ associated with a μ-symmetric (sub-)Markovian transition function.

Definition A.2 Let $\{\mathcal{P}_t\}_{t\in(0,\infty)}$ be a (sub-)Markovian transition function on $(E,\mathcal{B}(E))$.
(1) $\{\mathcal{P}_t\}_{t\in(0,\infty)}$ is called *conservative* if and only if $\mathcal{P}_t(x,E) = 1$ for any $(t,x)\in(0,\infty)\times E$.
(2) We say that $\{\mathcal{P}_t\}_{t\in(0,\infty)}$ has the *Feller property*, or simply it is *Feller*, if and only if $\mathcal{P}_t(C_\infty(E)) \subset C_\infty(E)$ for any $t\in(0,\infty)$ and $\lim_{t\downarrow 0}\|\mathcal{P}_t u - u\|_\infty = 0$ for any $u\in C_\infty(E)$.
(3) We say that $\{\mathcal{P}_t\}_{t\in(0,\infty)}$ has the *strong Feller property*, or simply it is *strong Feller*, if and only if $\mathcal{P}_t u \in C_b(E)$ for any bounded Borel measurable $u : E\to\mathbb{R}$.

The following proposition provides a sufficient condition for (CHK) and for strict positivity of the jointly continuous heat kernel.

Proposition A.3 *Assume $\mu(E) < \infty$ and suppose that $\{T_t\}_{t\in(0,\infty)}$ is ultracontractive. Let $\{\mathcal{P}_t\}_{t\in(0,\infty)}$ be a μ-symmetric strong Feller (sub-)Markovian transition function on $(E,\mathcal{B}(E))$ whose associated Markovian semigroup on $L^2(E,\mu)$ is $\{T_t\}_{t\in(0,\infty)}$. Then*
(1) (CHK) *holds with jointly continuous heat kernel $\{p_t\}_{t\in(0,\infty)} \subset C_b(E\times E)$, and $\mathcal{P}_t(x,A) = \int_A p_t(x,y)d\mu(y)$ for any $(t,x)\in(0,\infty)\times E$ and any $A\in\mathcal{B}(E)$.*
(2) Suppose that E is arcwise connected and that there exists a Hunt process $X = \bigl(\Omega,\mathcal{M},\{X_t\}_{t\in[0,\infty]},\{\mathbf{P}_x\}_{x\in E_\Delta}\bigr)$ on E whose transition function is $\{\mathcal{P}_t\}_{t\in(0,\infty)}$. Then $p_t(x,y)\in(0,\infty)$ for any $(t,x,y)\in(0,\infty)\times E\times E$. In particular, $(\mathcal{E},\mathcal{F})$ is irreducible.

Proof. (1) Let $\varphi\in\mathcal{D}[H]$ and $\lambda\in[0,\infty)$ satisfy $H\varphi = \lambda\varphi$. By the ultracontractivity of $\{T_t\}_{t\in(0,\infty)}$, $T_1\varphi = e^{-\lambda}\varphi \in L^\infty(E,\mu)$, and we may choose a bounded Borel measurable version of φ. Since $\varphi = e^{\lambda}T_1\varphi = e^{\lambda}\mathcal{P}_1\varphi$ μ-a.e. on E and $\mathcal{P}_1\varphi\in C_b(E)$ by the strong Feller property of $\{\mathcal{P}_t\}_{t\in(0,\infty)}$, we may assume that $\varphi\in C_b(E)$. Now as in [14, Proof of Theorem 2.1.4], for any $T\in(0,\infty)$, the eigenfunction expansion [14, (2.1.4)] of the heat kernel defines an absolutely norm-convergent series in the Banach space $C_b([T,\infty)\times E\times E)$. Hence the heat kernel $\{p_t\}_{t\in(0,\infty)}$ defined by [14, (2.1.4)] is jointly continuous, proving (CHK). Moreover, if $t\in(0,\infty)$ and $A\in\mathcal{B}(E)$ then $\mathcal{P}_t\boldsymbol{1}_A, \int_A p_t(\cdot,y)d\mu(y)\in C_b(E)$ and both of them are equal to $T_t\boldsymbol{1}_A$ μ-a.e. on E, hence they are equal at every point of E.
(2) Suppose $p_t(x,x) = 0$ for some $(t,x)\in(0,\infty)\times E$. Then $p_{t/2}(x,y) = 0$ for any $y\in E$ since $0 = p_t(x,x) = \int_E p_{t/2}(x,z)^2 d\mu(z)$. Inductively, for each $n\in\mathbb{N}$, $p_{t/2^n}(x,y) = 0$ for any $y\in E$ and hence $\mathbf{P}_x[X_{t/2^n}\in E] = \mathcal{P}_{t/2^n}(x,E) = \int_E p_{t/2^n}(x,y)d\mu(y) = 0$. Therefore $X_{t/2^n} = \Delta_E$ for any $n\in\mathbb{N}$ $\mathbf{P}_x$-a.s., which then implies that $X_0 = \Delta_E$ $\mathbf{P}_x$-a.s. since $X_{t/2^n}(\omega)\xrightarrow{n\to\infty} X_0(\omega)$ in E_Δ for any $\omega\in\Omega$. This contradicts $\mathbf{P}_x[X_0 = x] = 1$. Therefore $p_t(x,x)\in(0,\infty)$ for any $(t,x)\in(0,\infty)\times E$. Now based on the arcwise connectivity of E, the positivity of p_t follows in exactly the same way as in [28, Proof of Theorem A.4]. Finally, for $A\in\mathcal{B}(E)$, $\bigl(\int_{(E\setminus A)\times A} p_t d(\mu\times\mu) =\bigr)\int_{E\setminus A} T_t\boldsymbol{1}_A d\mu = 0$ implies $\mu\times\mu((E\setminus A)\times A) = \mu(E\setminus A)\mu(A) = 0$, which is sufficient for $(\mathcal{E},\mathcal{F})$ to be irreducible, by [17, pp.46-48, Section 1.6]. ■

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In the theorem below, we deduce a uniform positivity of short time hitting probabilities by assuming the positivity of capacity. Recall the following definitions.

Definition A.4 (1) A closed subset F of E is called *μ-regular* if and only if, for any open subset U of E, either $\mu(U \cap F) > 0$ or $U \cap F = \emptyset$.
(2) We define, with the convention that $\inf \emptyset := \infty$,

$$\mathrm{cap}_{\mathcal{E}}(U) := \inf\{\mathcal{E}_1(u,u) \mid u \in \mathcal{F}, u \geq 1\ \mu\text{-a.e. on } U\} \qquad \text{for } U \subset E \text{ open}, \tag{A.2}$$

$$\mathrm{Cap}_{\mathcal{E}}(A) := \inf\{\mathrm{cap}_{\mathcal{E}}(U) \mid U \subset E \text{ open}, A \subset U\} \qquad \text{for } A \subset E. \tag{A.3}$$

$\mathrm{Cap}_{\mathcal{E}}$ is clearly an extension of $\mathrm{cap}_{\mathcal{E}}$. Moreover, let $A \subset E$ and let $\mathscr{S}(x)$ be a statement on x for each $x \in A$. Then we say that *$\mathscr{S}$ holds $\mathcal{E}$-q.e. on A* if and only if $\mathrm{Cap}_{\mathcal{E}}\bigl(\{x \in A \mid \mathscr{S}(x) \text{ fails}\}\bigr) = 0$. When $A = E$ we simply say *'$\mathscr{S}$ holds $\mathcal{E}$-q.e.'* instead.

Theorem A.5 *Let $X = \bigl(\Omega, \mathcal{M}, \{X_t\}_{t \in [0,\infty]}, \{\mathbf{P}_x\}_{x \in E_\Delta}\bigr)$ be a μ-symmetric Hunt process on E whose Dirichlet form on $L^2(E,\mu)$ is $(\mathcal{E},\mathcal{F})$. For $A \in \mathcal{B}(E_\Delta)$ and $\omega \in \Omega$, define*

$$\sigma_A(\omega) := \inf\{t \in [0,\infty) \mid X_t(\omega) \in A\} \qquad (\inf \emptyset := \infty). \tag{A.4}$$

If $A \in \mathcal{B}(E)$ and $\mathrm{Cap}_{\mathcal{E}}(A) \in (0,\infty)$, then there exists a μ-regular closed subset F of E with the following properties: $A \cap F \neq \emptyset$, and for any $x_0 \in A \cap F$, any $t \in (0,\infty)$ and any $s \in (0,1)$ there exists an open neighborhood U of x_0 in E such that

$$\inf_{x \in U \cap F} \mathbf{P}_x[\sigma_A \leq t] \geq s. \tag{A.5}$$

Proof. Let $\sigma_A^+(\omega) := \inf\{t \in (0,\infty) \mid X_t(\omega) \in A\}$ $(\inf \emptyset := \infty)$ for $\omega \in \Omega$, and set $N_A := \bigl\{x \in E \mid \mathbf{P}_x[\sigma_A = \sigma_A^+] \neq 1\bigr\}$. Then $\mathrm{Cap}_{\mathcal{E}}(N_A) = 0$ by [17, Theorems 4.1.3, 4.2.1 (ii) and A.2.6 (i)]. Let $p_A^1(x) := \mathbf{E}_x[e^{-\sigma_A}]$ and $p_A^{1+}(x) := \mathbf{E}_x\bigl[e^{-\sigma_A^+}\bigr]$ for $x \in E$. Then $p_A^1 = p_A^{1+}$ on $E \setminus N_A$. Since p_A^{1+} is quasi-continuous by [17, Theorem 4.2.5] and $\mathrm{Cap}_{\mathcal{E}}(N_A) = 0$, p_A^1 is also quasi-continuous. By [17, Theorem 2.1.2 (i)] there exists a μ-regular closed subset F of E such that $\mathrm{Cap}_{\mathcal{E}}(A) > \mathrm{Cap}_{\mathcal{E}}(E \setminus F)$ and $\bigl(p_A^1\bigr)|_F$ is continuous.

This F possesses the required properties. Indeed, $A \cap F \neq \emptyset$ follows from $\mathrm{Cap}_{\mathcal{E}}(A) > \mathrm{Cap}_{\mathcal{E}}(E \setminus F)$. Let $x_0 \in A \cap F$, $t \in (0,\infty)$ and $s \in (0,1)$ and set $M_{s,t} := s + (1-s)e^{-t}$ (< 1). Since $\bigl(p_A^1\bigr)|_F$ is continuous and $p_A^1(x_0) = 1$, we may choose an open neighborhood U of x_0 in E so that $p_A^1(x) \geq M_{s,t}$ for any $x \in U \cap F$. Now let $x \in U \cap F$. Then since

$$\begin{aligned} M_{s,t} \leq p_A^1(x) &= \mathbf{E}_x[e^{-\sigma_A}] = \mathbf{E}_x[e^{-\sigma_A}\mathbf{1}_{\{\sigma_A \leq t\}}] + \mathbf{E}_x[e^{-\sigma_A}\mathbf{1}_{\{\sigma_A > t\}}] \\ &\leq \mathbf{P}_x[\sigma_A \leq t] + e^{-t}\mathbf{P}_x[\sigma_A > t] = \mathbf{P}_x[\sigma_A \leq t] + e^{-t}(1 - \mathbf{P}_x[\sigma_A \leq t]), \end{aligned}$$

we conclude that $\mathbf{P}_x[\sigma_A \leq t] \geq (M_{s,t} - e^{-t})/(1 - e^{-t}) = s$. Therefore (A.5) follows. ■

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